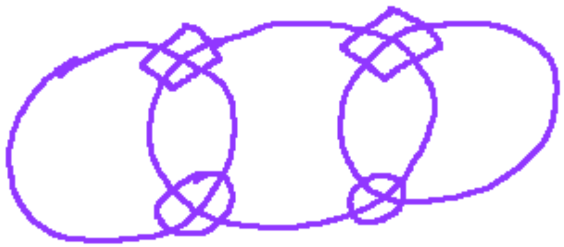
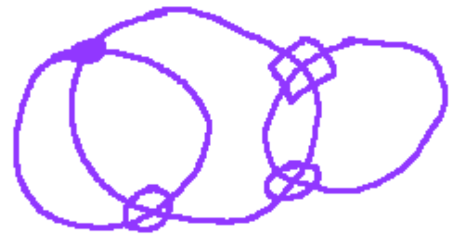
Any form of loops with the two types of virtual crossing then has a chromatic evaluation.

e.g.  $= 2$  $-$ 

$= 2$  $- \bigcirc$ 

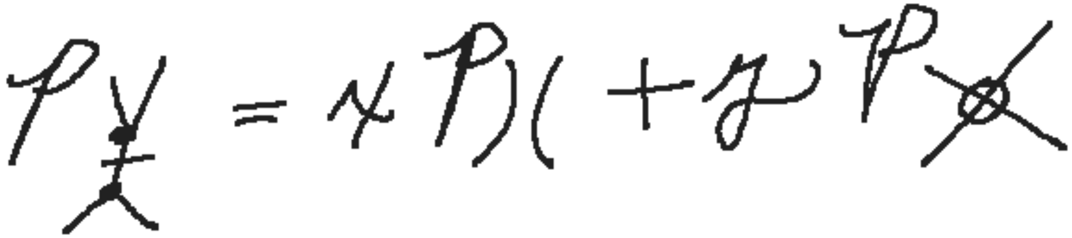
$= (2-\delta)$  $= (2-\delta)(2\delta - \delta^2)$

We can define this chromatic evaluation via nodal colors

by $\mathcal{C}$  $= 2\,\mathcal{C}$  $- \mathcal{C}$ 

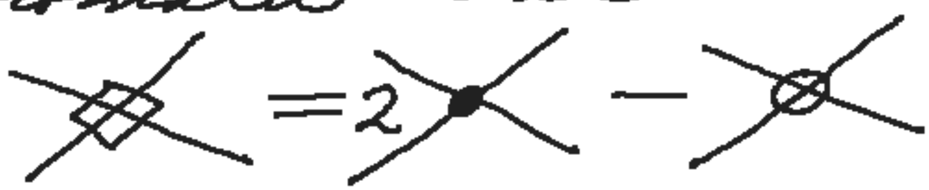
and so it is a contraction/deletion algorithm.

Now we have a generalized Penrose perfect matching polynomial.

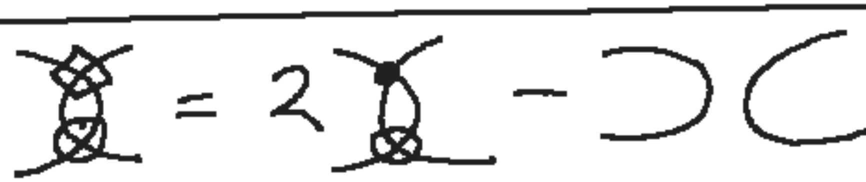


$$P_{\bigcirc} = \delta$$

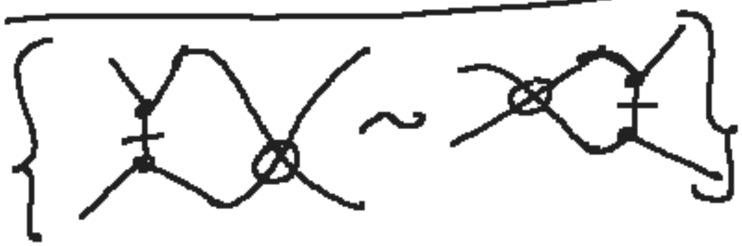
In context of double virtual chromatic evaluation



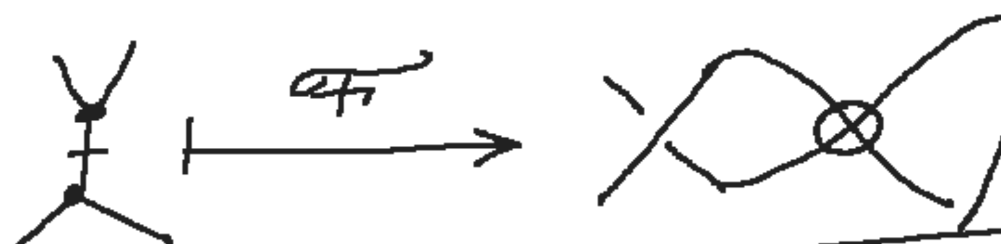
<u>Note</u>:



$$\left\{\begin{array}{l}\text{Trivalent}\\ \text{Double Virtual}\\ \text{Graphs}\\ \text{with PM}\end{array}\right\} \xrightarrow{\ F\ } \left\{\begin{array}{l}\text{Double}\\ \text{Virtual}\\ \text{Knots \& Links}\end{array}\right\}$$



⟨⟨⟩⟩ can be removed by adding orientations



F sends P to an extended bracket.

Then $P_{\times} = P_{\ldots} = x P_{\ldots} + y P_{\ldots}$

$$\boxed{P_{\times} = x P_{)(} + y P_{\asymp}}$$

$P_{O} = \delta$

$\neq$ context

Virtual Knot Theory

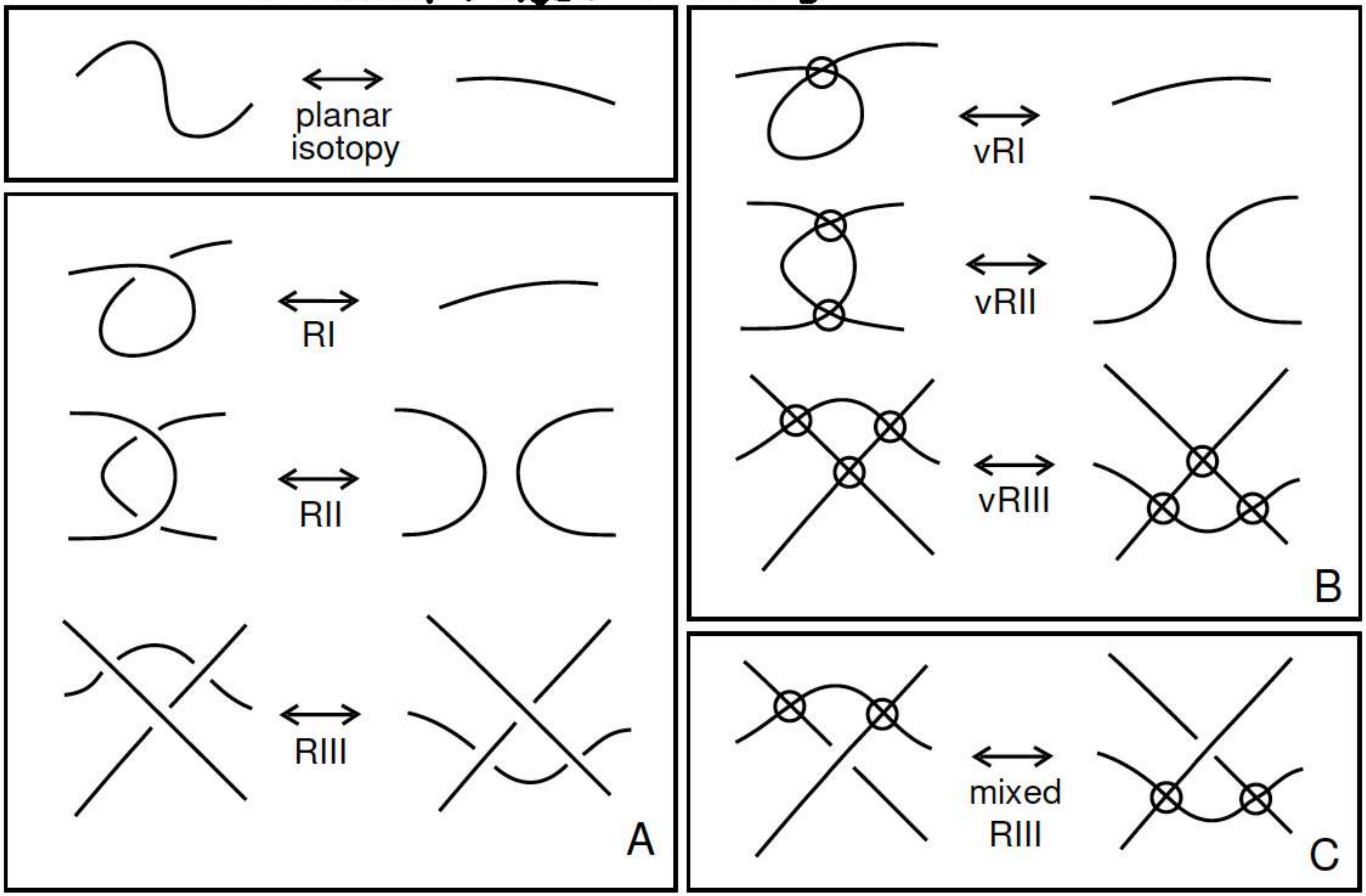


Figure 27: **Moves**

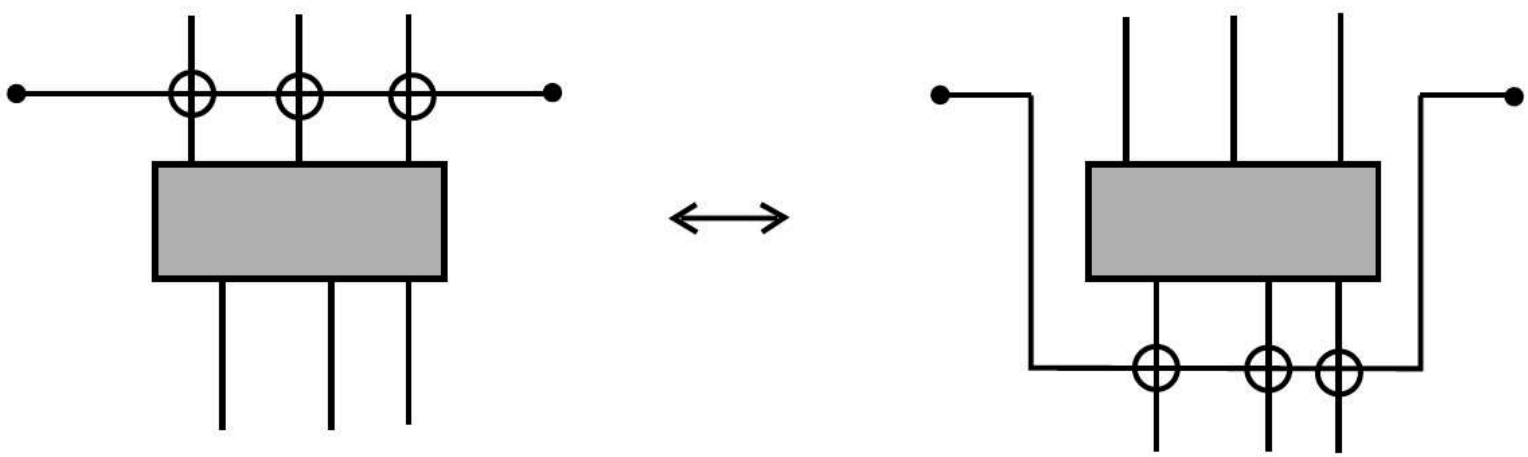


Figure 28: **Detour Move**

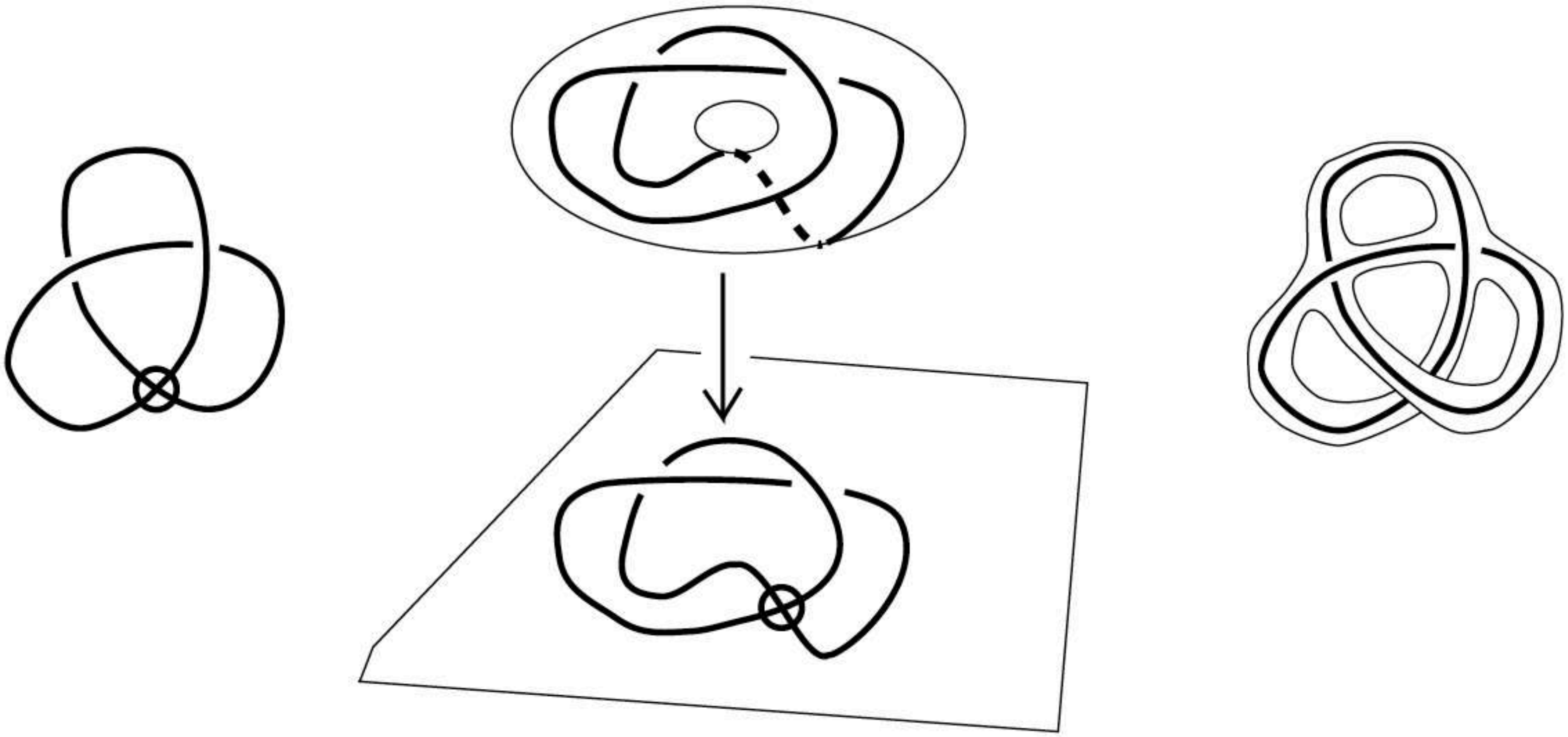


Figure 30: **Surfaces and Virtuals**

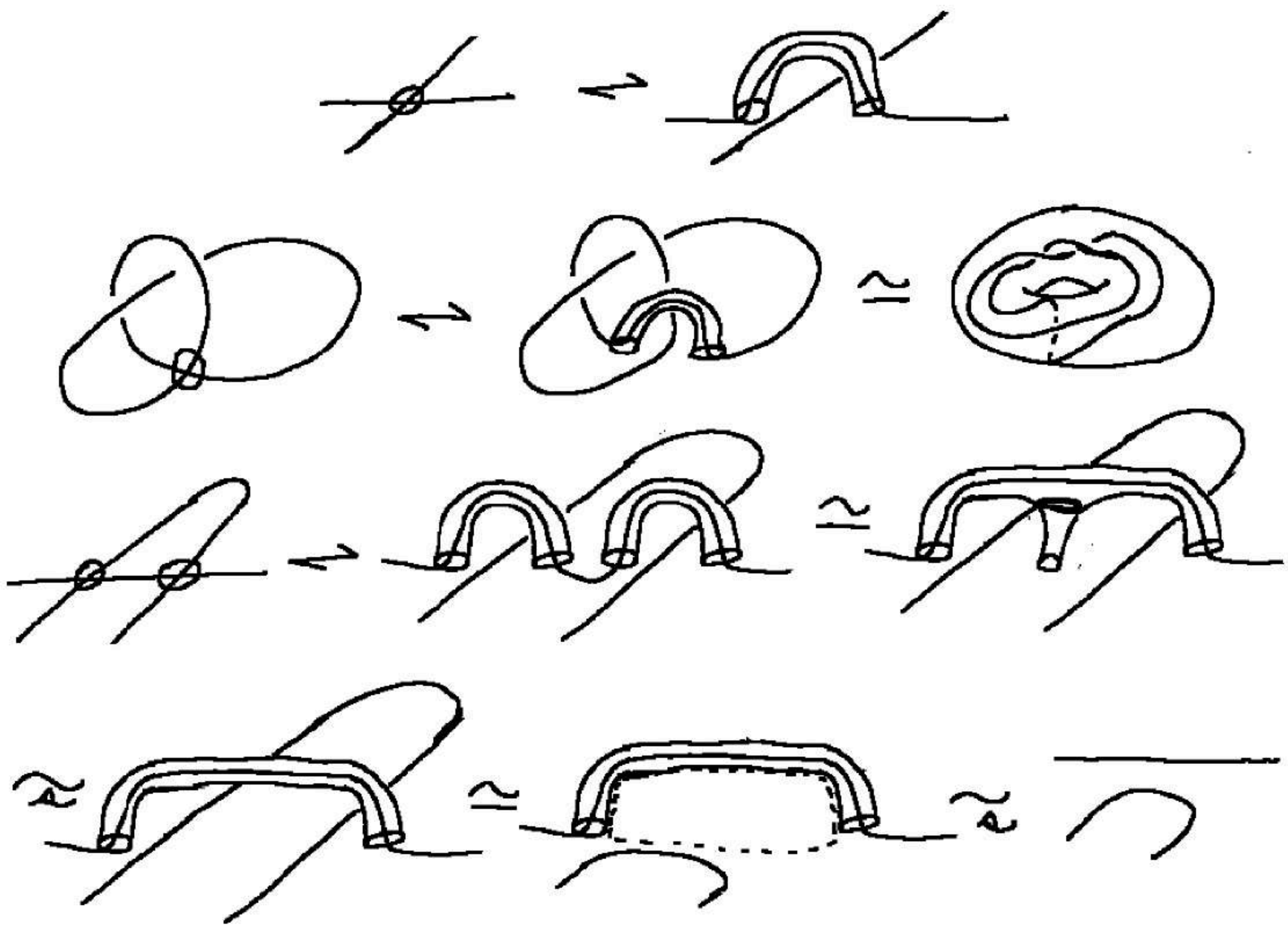
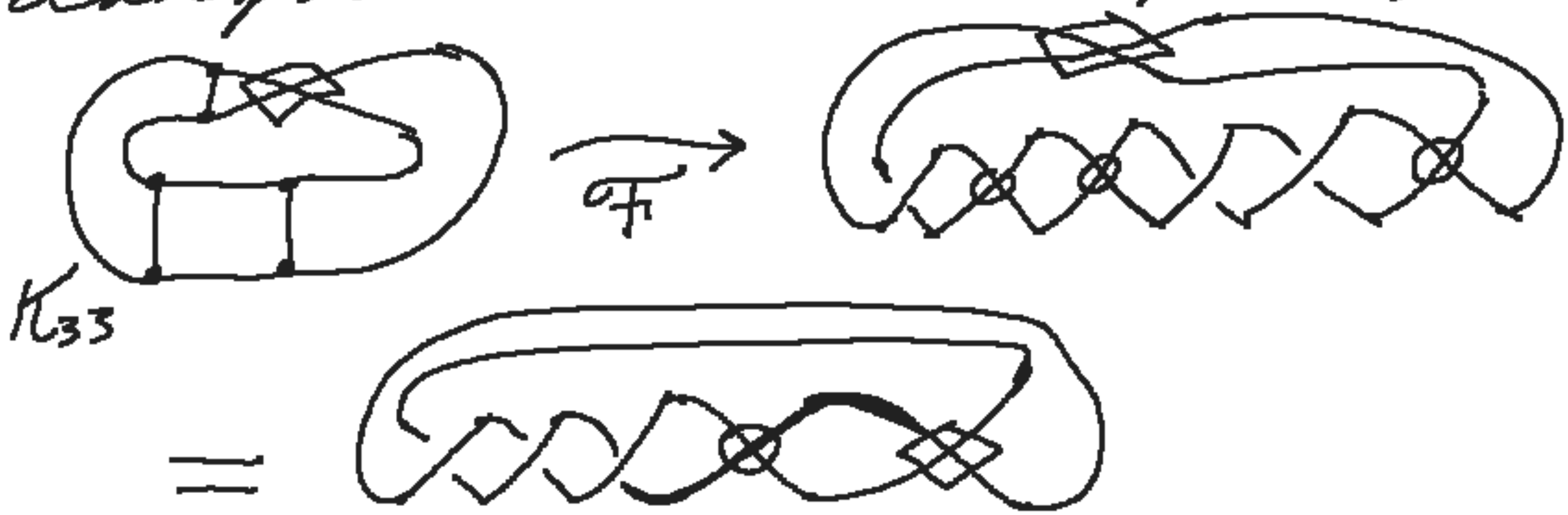


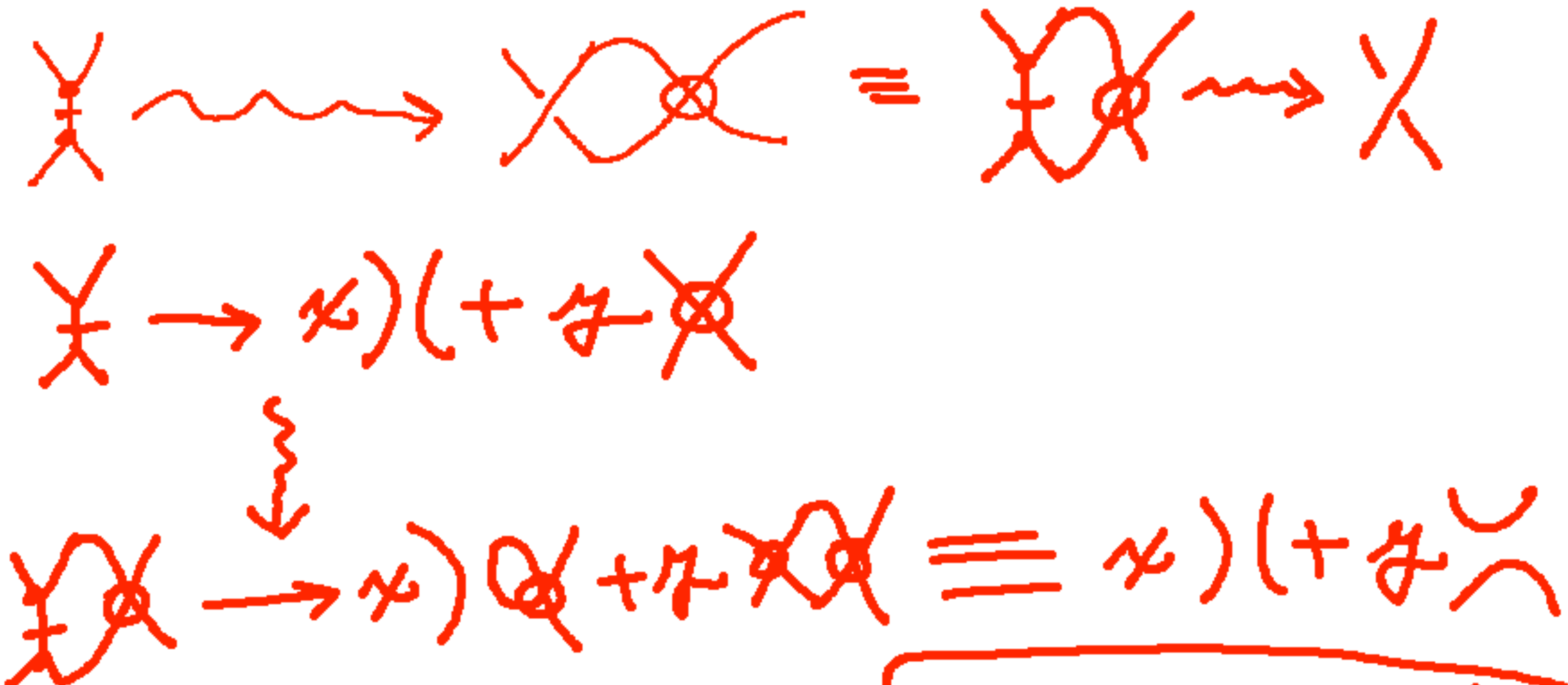
Figure 31: **Replacing Virtual Crossings by Handle Detours**

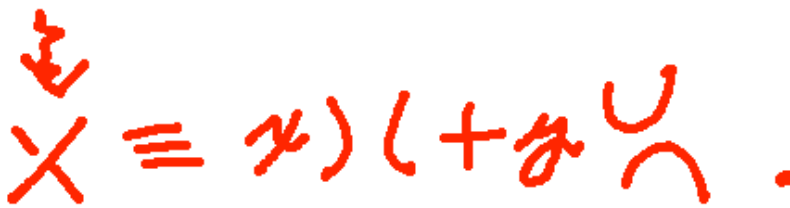
It is of interest to go back and forth. For example,



and this is an example of a virtual knot whose topological type is influenced by the doubling.

Transition to Virtual Knot Theory





Penrose Gen Poly

$\rightsquigarrow$

Gen Bracket Poly

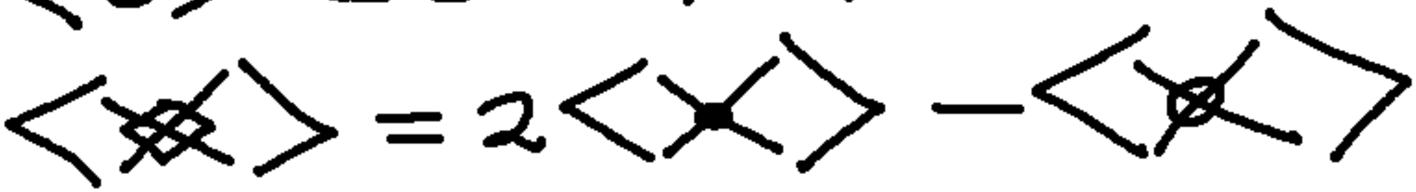
Thus we will have
<u>multi-virtual knot theory</u>
with ,  (and α, $\alpha \in$ SomeSet)

- each virtual crossing detours over all other virtual crossings (and over classical crossings).

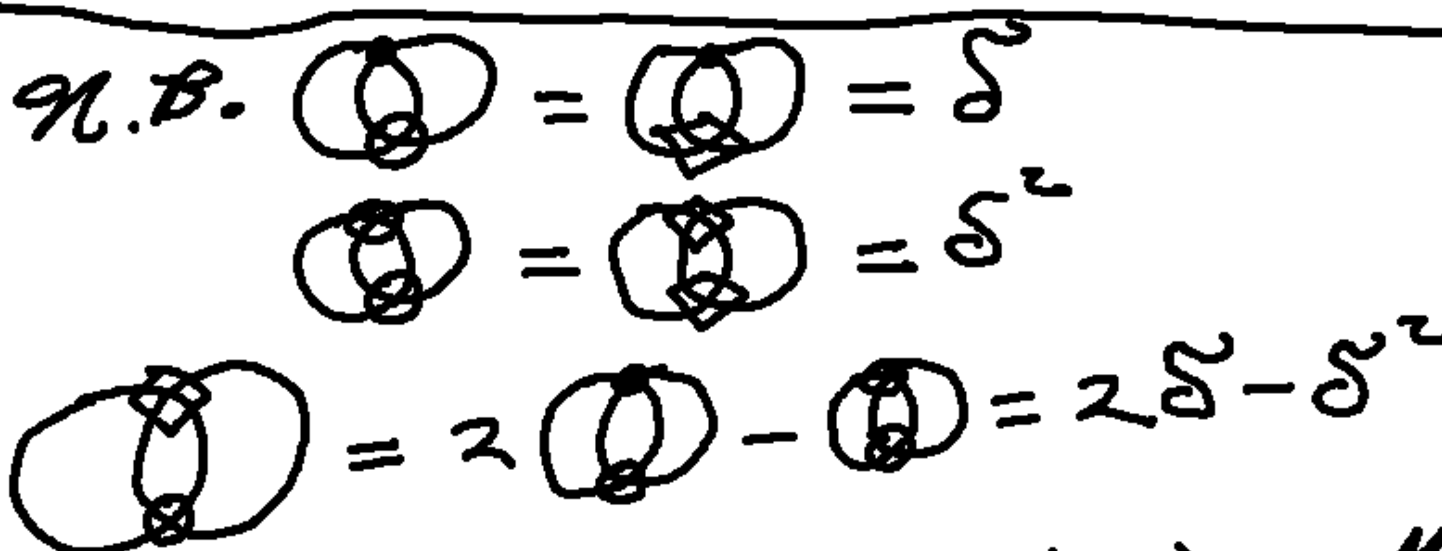
-  this does <u>not</u> reduce.

Generalized MV Bracket

$\langle \bigcirc \rangle = \delta = -A^2 - A^{-2}$



N.B.



Thm. This gives an MV invariant.

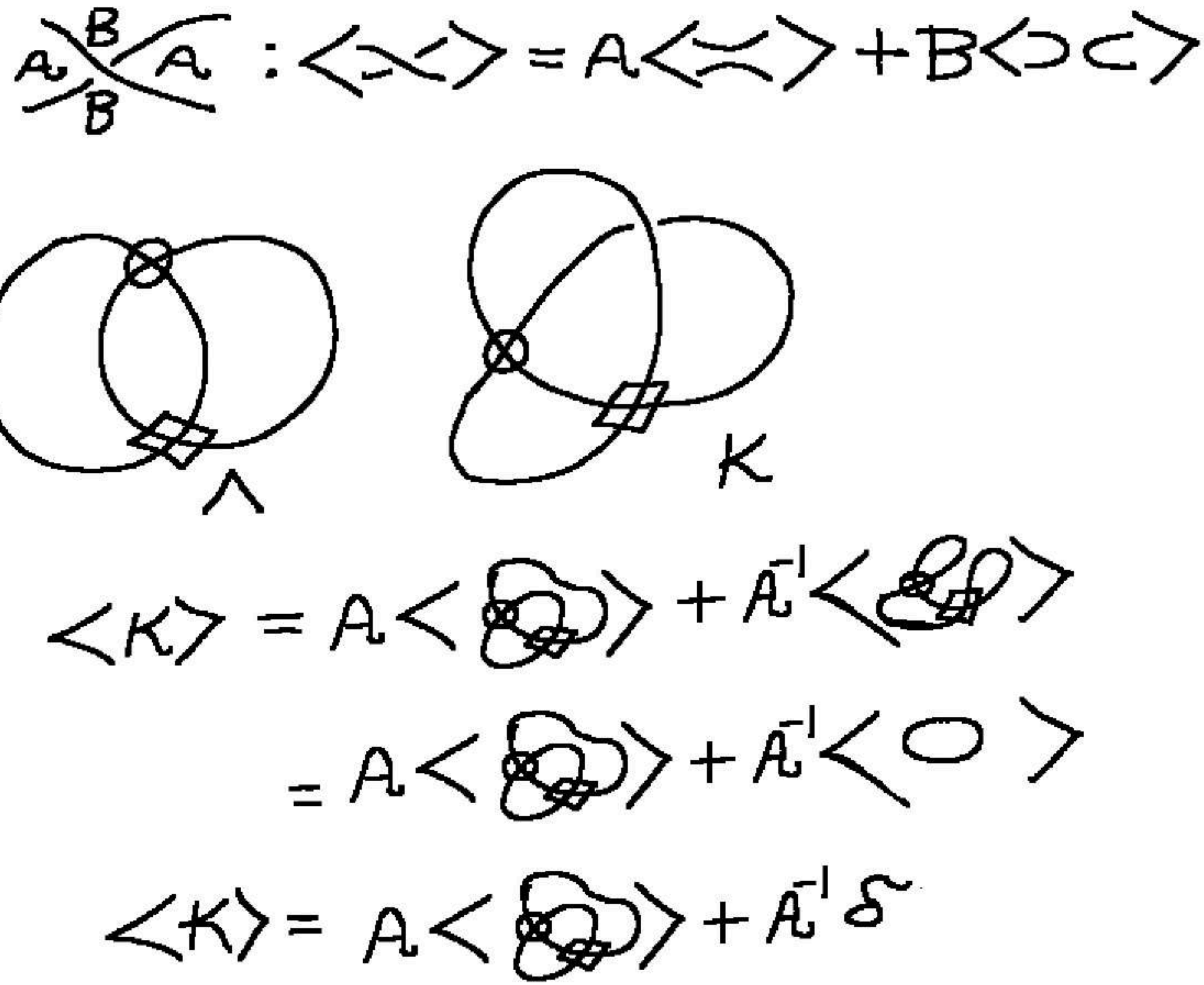


Figure 33: **Double Virtual Link and Double Virtual Knot**

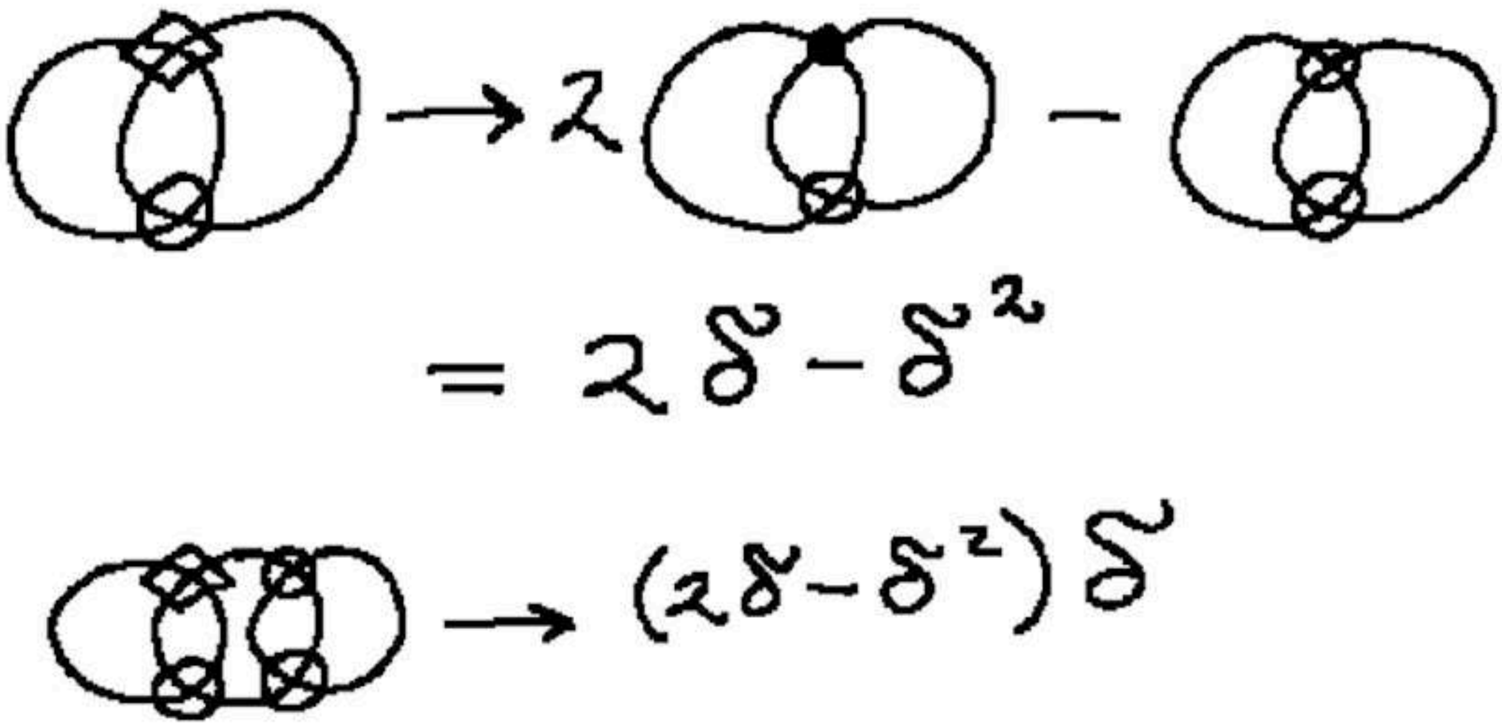


Figure 40: **Loop Evaluations**

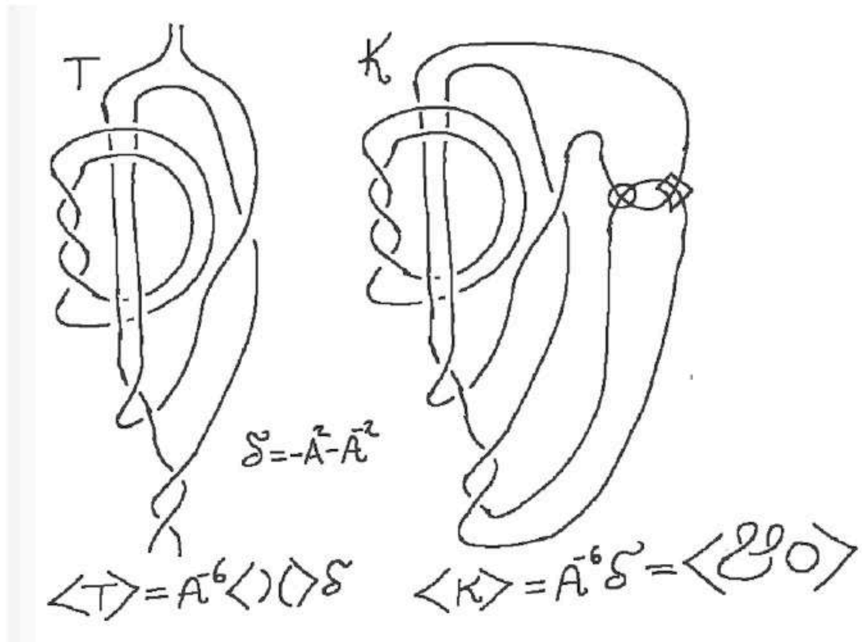
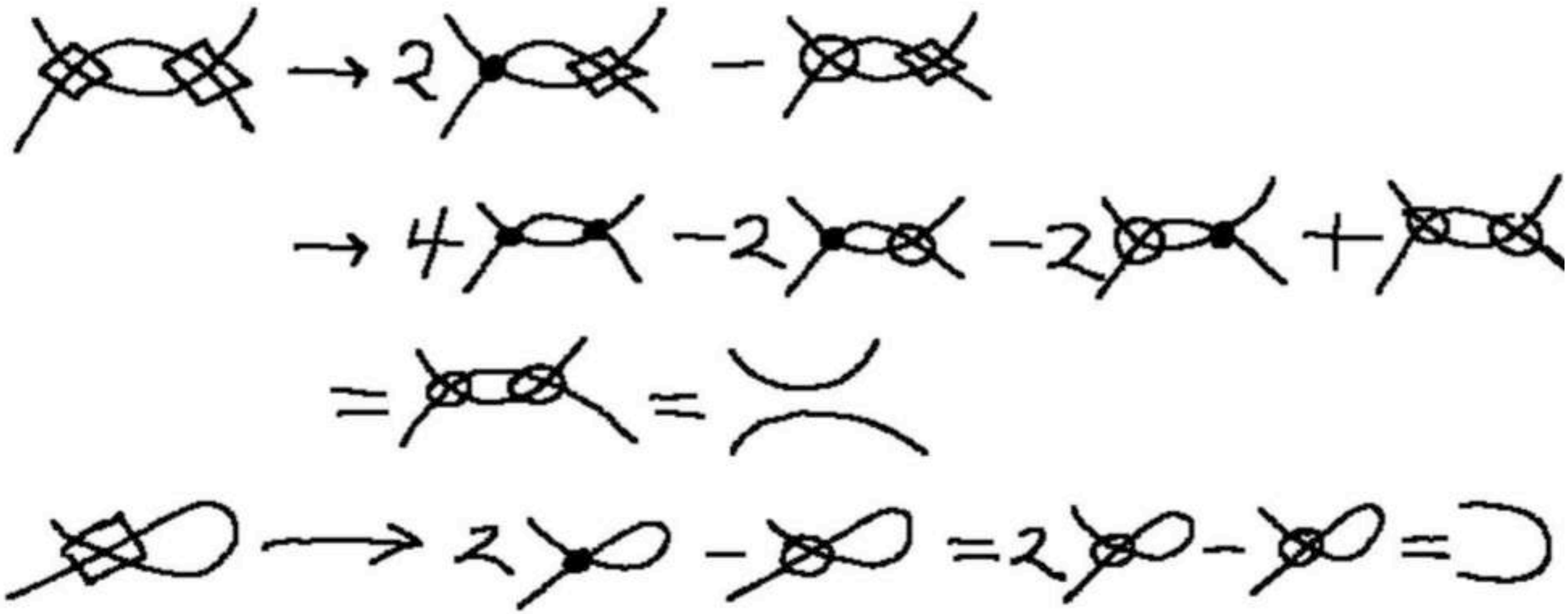
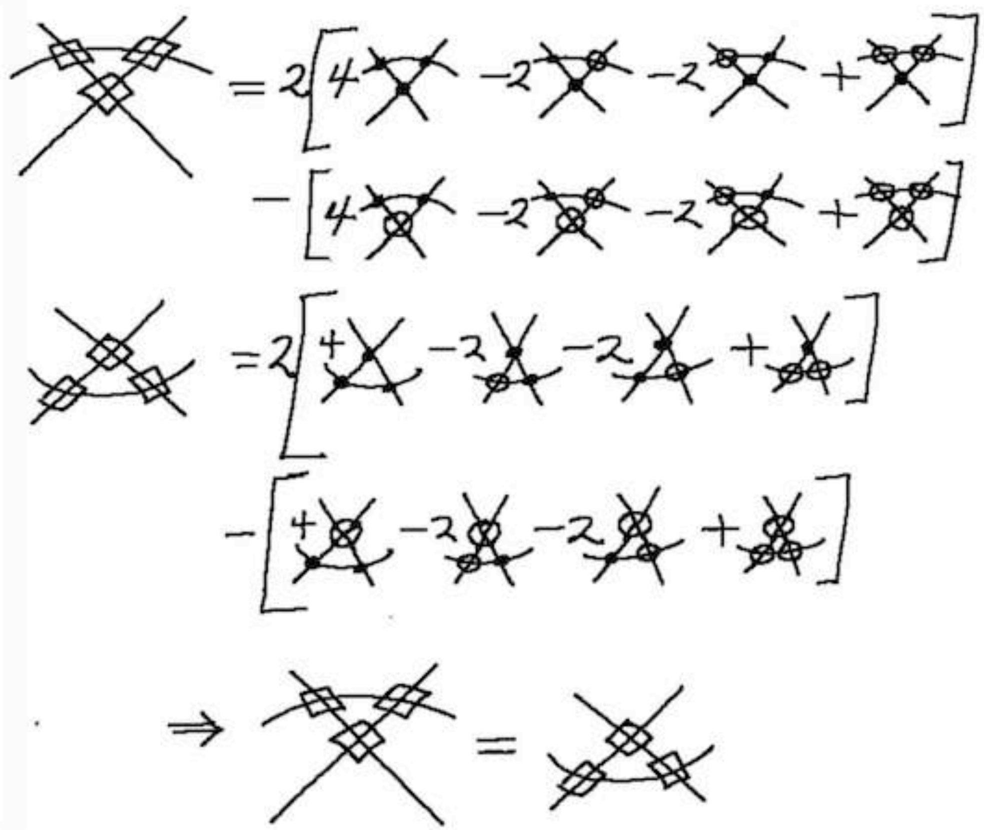


Figure 49: **Example of Non-Trivial Double Virtual Knot whose Virtuality is Invisible to Generalized Bracket**





 $= A$  $+ A^{-1}$ 

$= A$  $+ A^{-1}\delta$

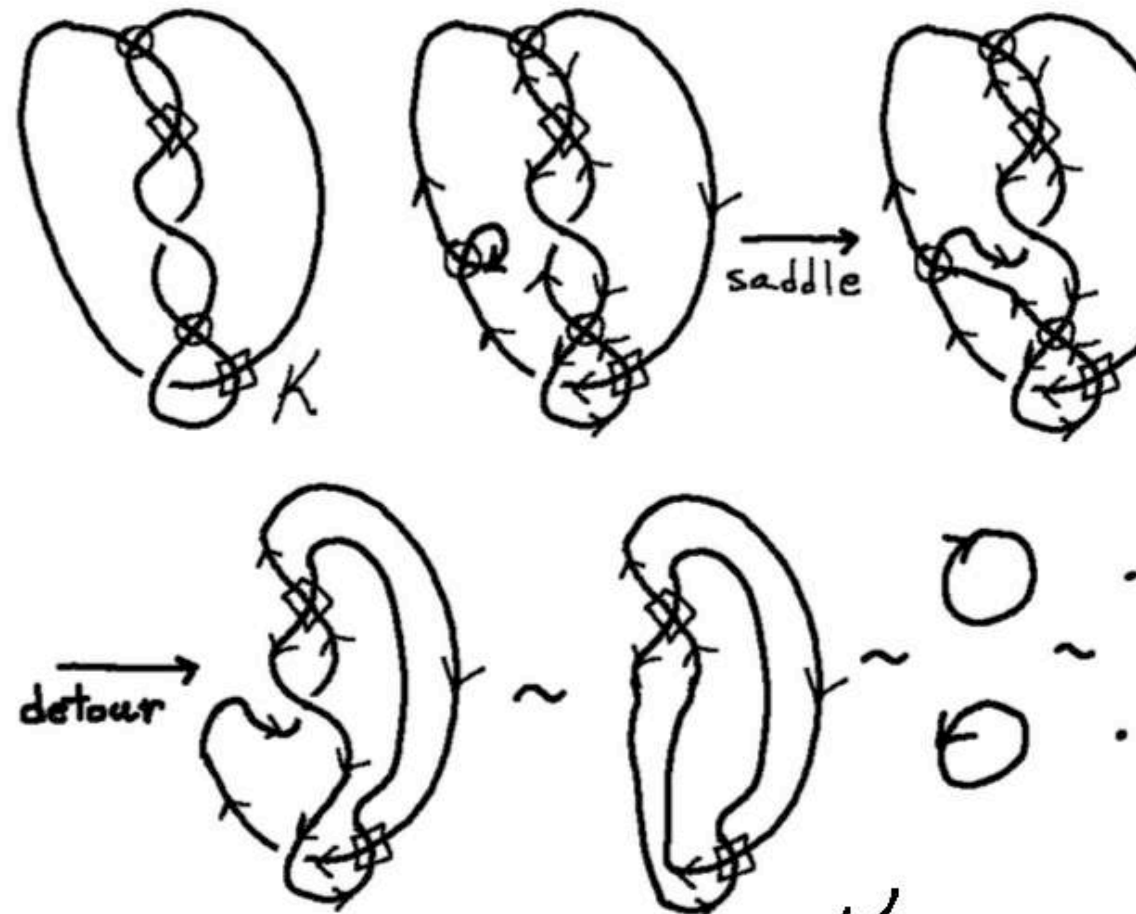
and in evaluating this generalized bracket, we take  $= 2\delta - \delta^2$.

One can leave virtual graphs in an evaluation.

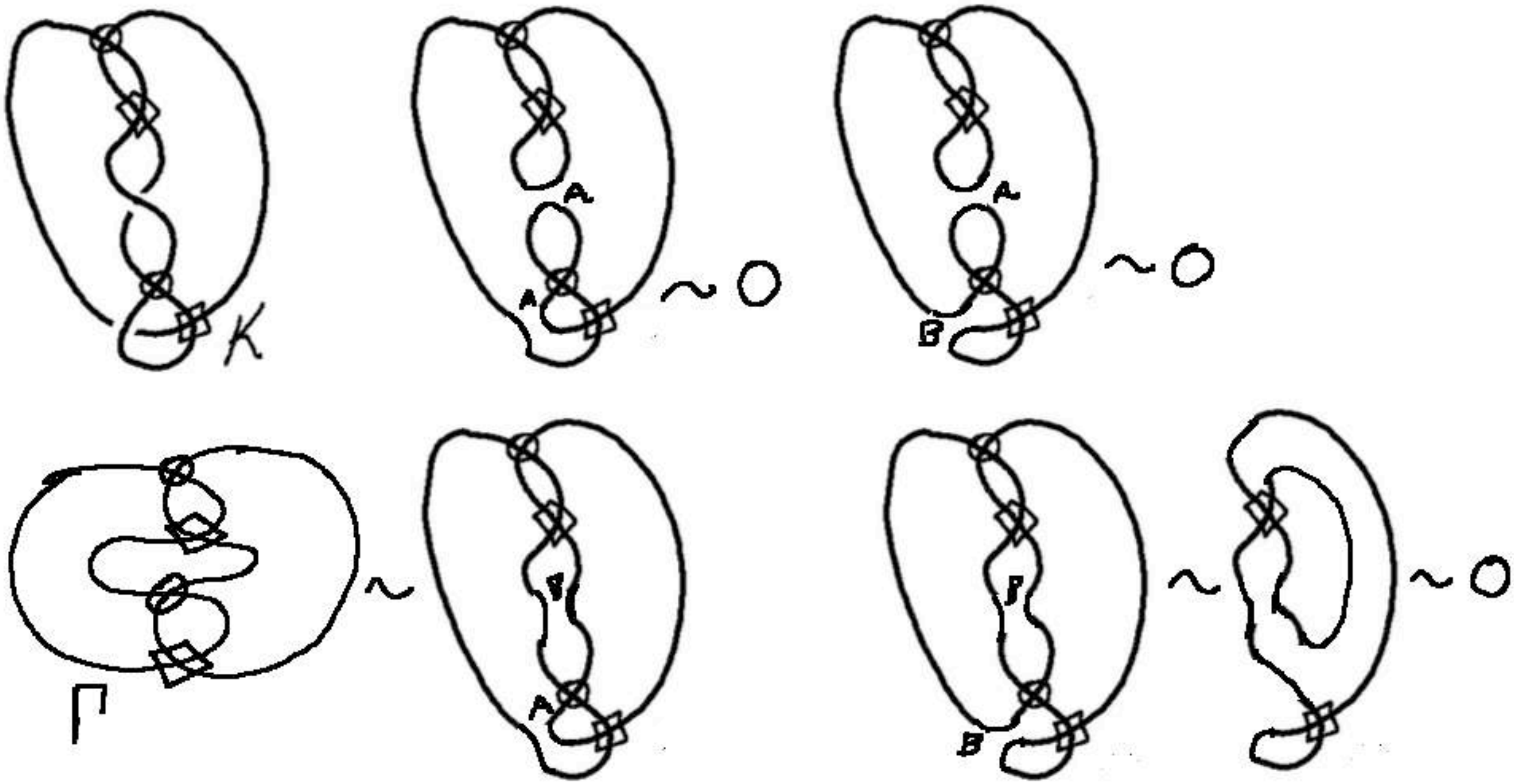
e.g.  is non-trivial but not detected by this method.

Here is an example. K is a slice knot in MV category.

(max, min, saddles + MV isotopy)



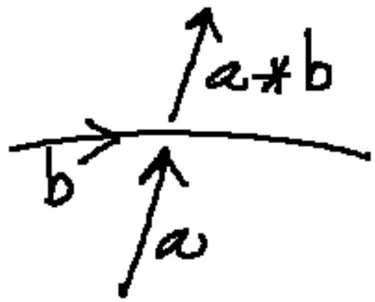
So we want to show that K is a non-trivial MV knot.



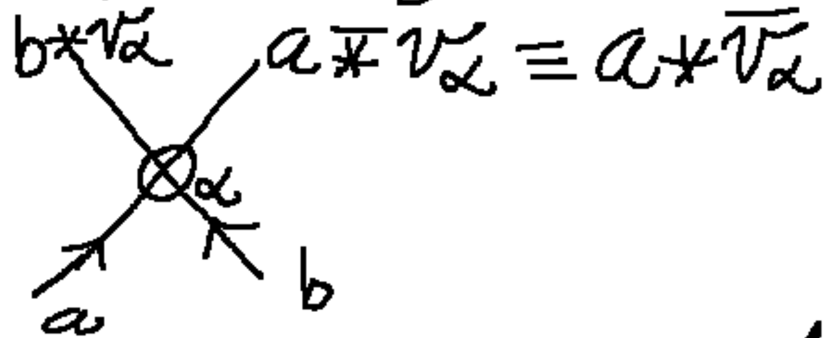
$\Rightarrow \langle K \rangle = A^2\delta + 2\delta + \Gamma$

The $\otimes = z \times - \otimes$ does not distinguish Γ from OO & so does not distinguish K from O.

However, the quandle also has an MV generalization.



and



$\{v_\alpha\}$ free gens of quandle automorphisms

and s.t. $(a*v_\alpha)*v_\beta$

$= (a*v_\beta)*v_\alpha$

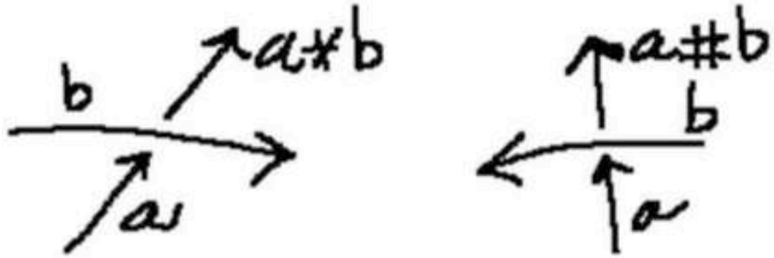
when $\alpha \neq \beta$.

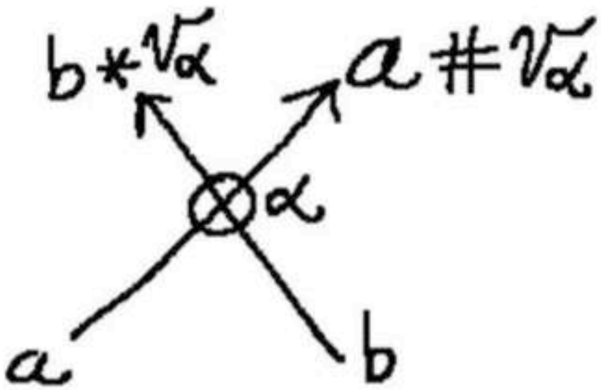
e.g.

$a*v_\alpha = v_\alpha a$

module elt in a Fox Alexander quandle.

Generalized Quandle



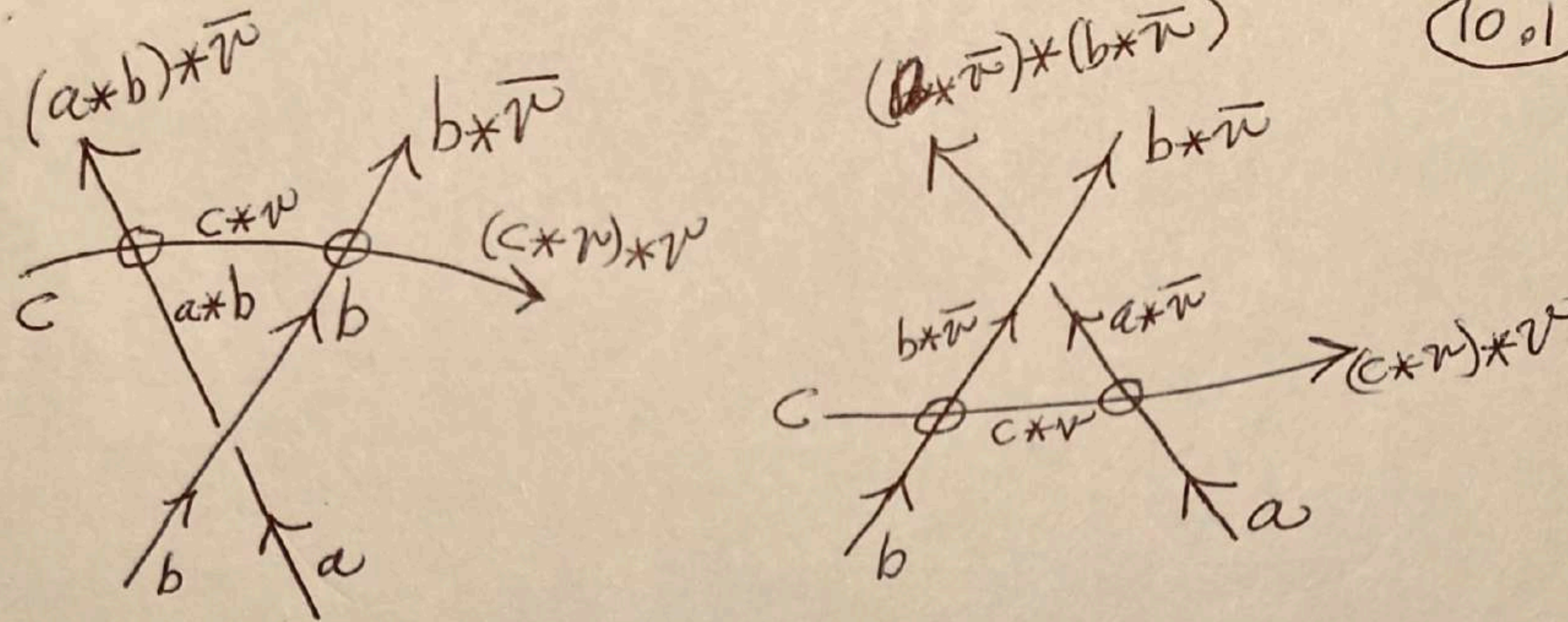


$$(x * v_\alpha) * v_\beta = (x * v_\beta) * v_\alpha$$

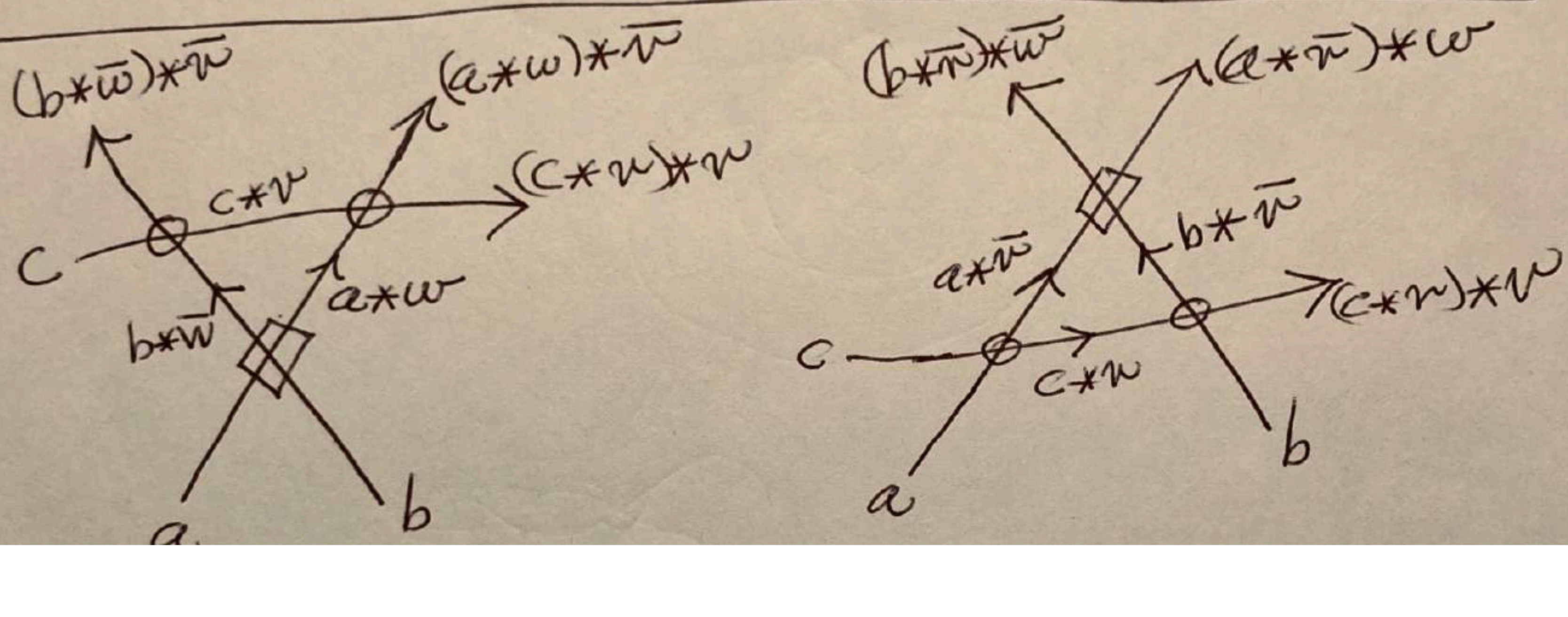
$$(x \# v_\alpha) \# v_\beta = (x \# v_\beta) \# v_\alpha$$

1. $a * a = a$, $a \# a = a$
2. $(a * b) \# b = a$, $(a \# b) * b = a$
3. $(a * b) * c = (a * c) * (b * c)$

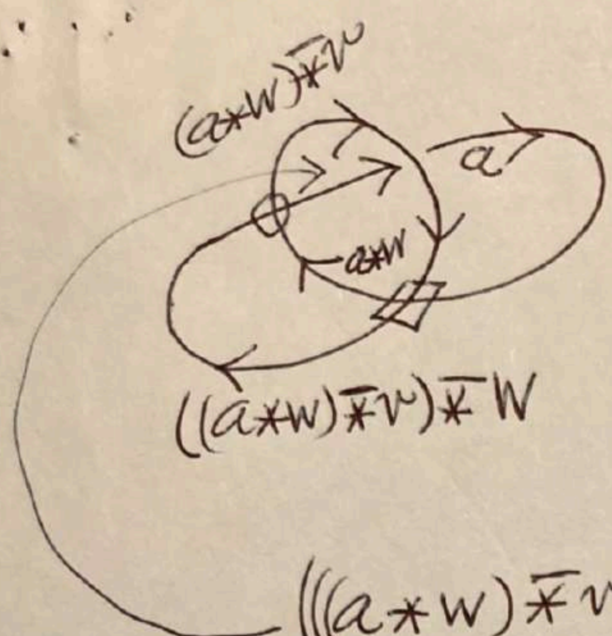
   $(a \# b) \# c = (a \# c) \# (b \# c)$

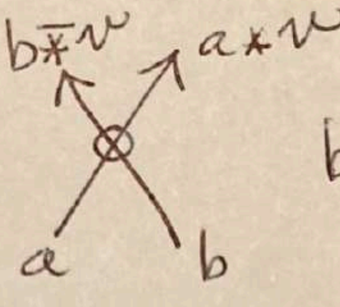


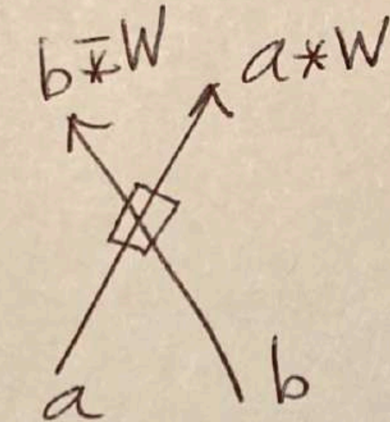
$$(a*b)*\overline{v} = (a*\overline{v})*(b*\overline{v})$$



11

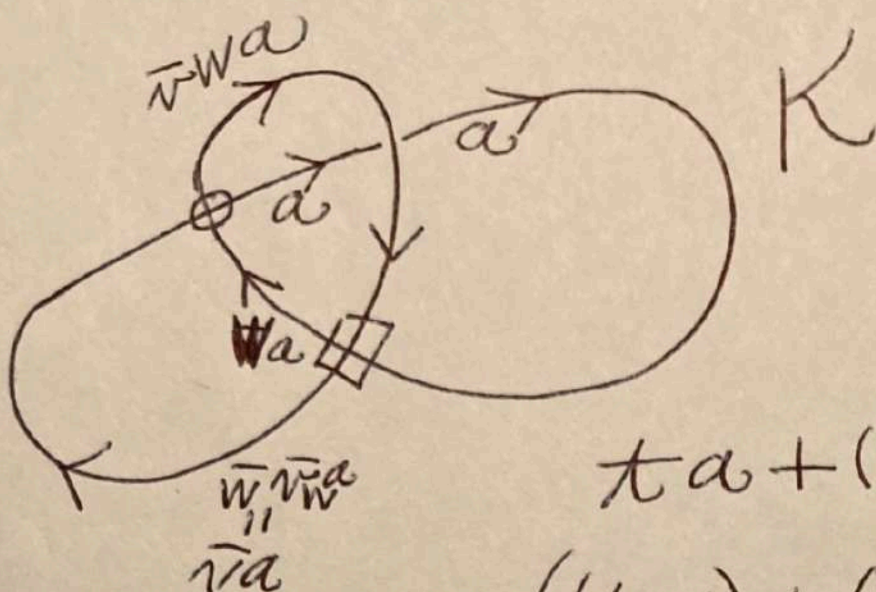






$$(((a*w)\bar{*}v)\bar{*}w)*v$$

$$a = \left[(((a*w)\bar{*}v)\bar{*}w)*v\right] * \left[(a*w)\bar{*}v\right]$$



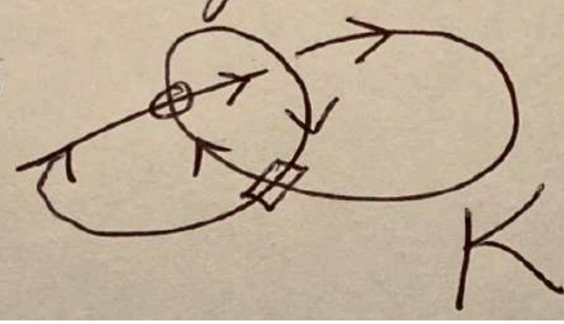
$$ta + (1-t)\bar{v}wa = a$$

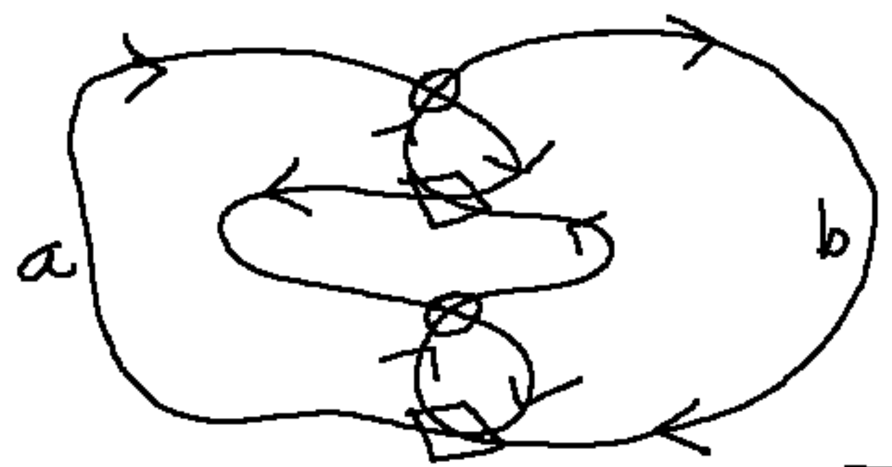
$$\big((t-1) + (1-t)\bar{v}w\big)a = 0$$

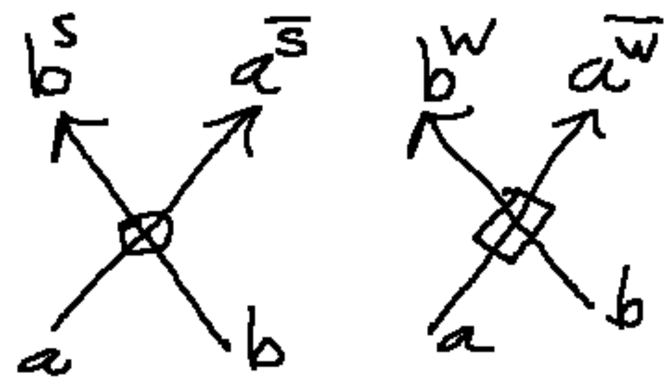
$$\underline{P(t,v,w) = (1-t)(1-\bar{v}w)}$$

Generalized Sawollek Poly

detects

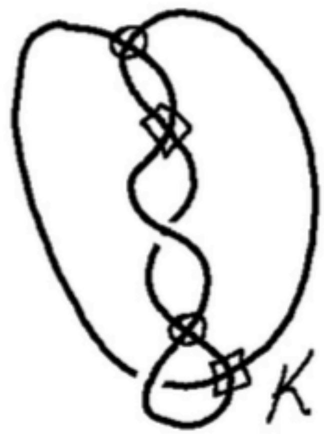




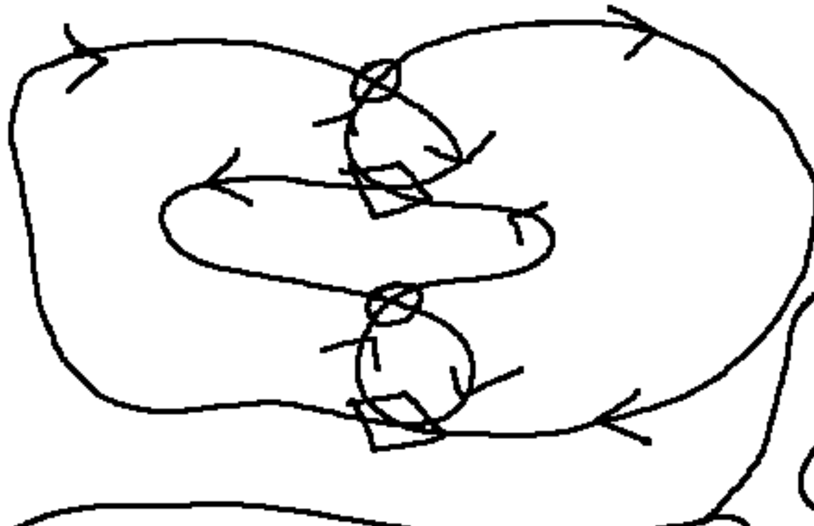


$$a = a^{\bar s w \bar s \bar w} \qquad b = b^{\bar w s \bar w s}$$

$\Rightarrow$ Π has non-trivial gen quandle

$\Rightarrow$ 

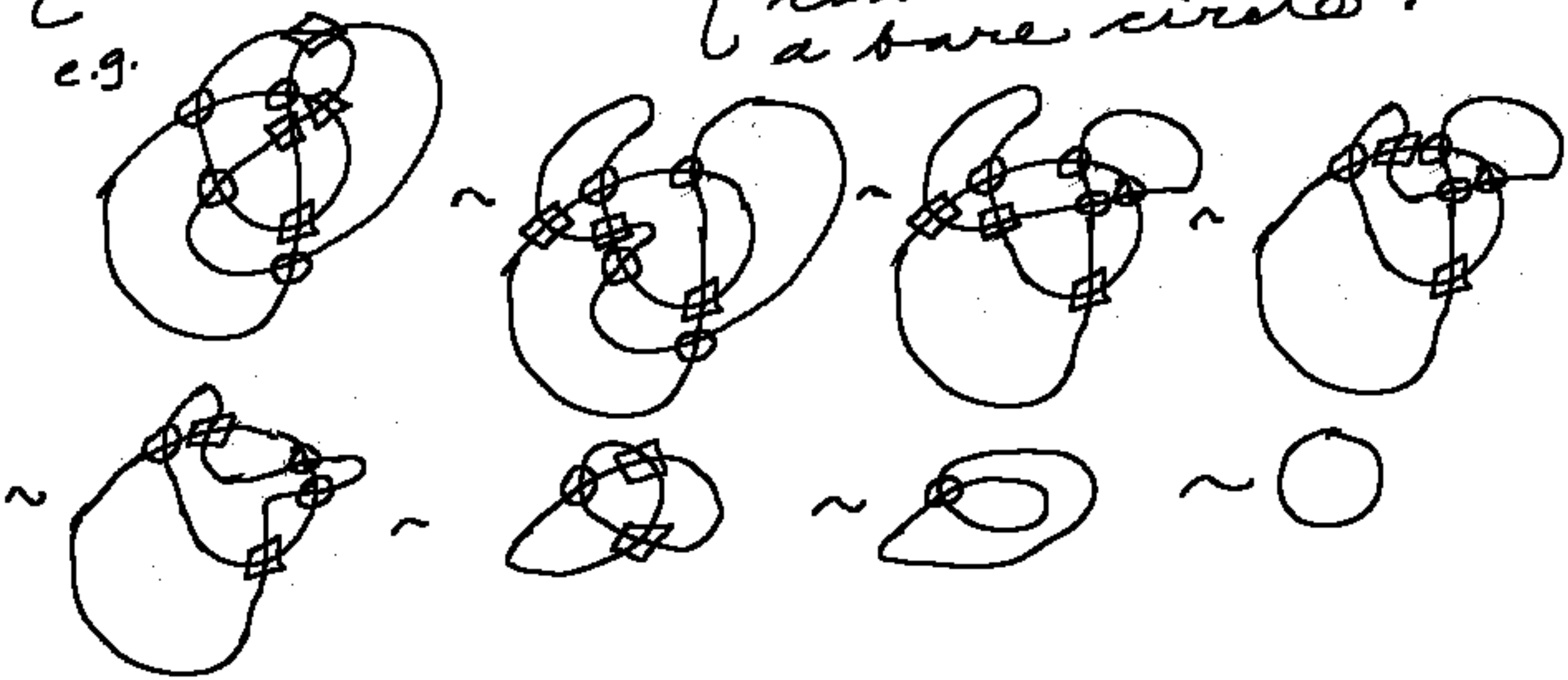
nontrivial
UV slice knot



We have shown that this multiflat is non-trivial.

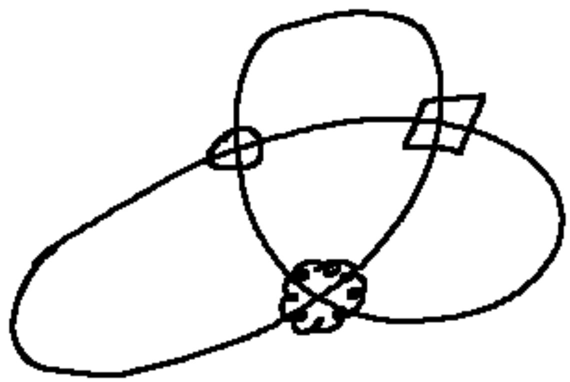
<u>Conjecture</u>. All single component two virtual curves reduce to a bare circle.

e.g.



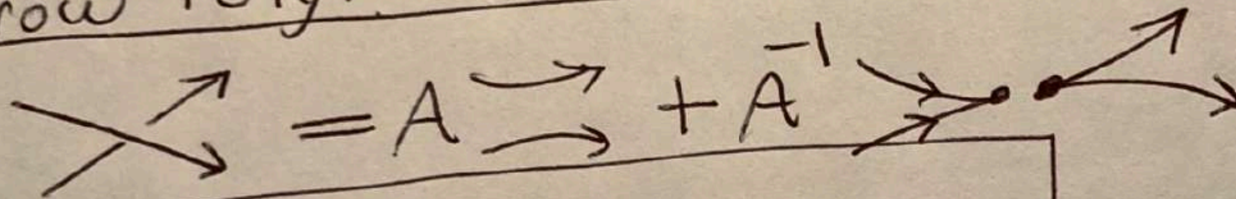
Conjecture. This is not trivial!

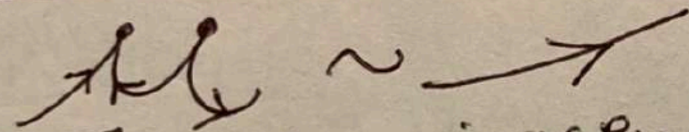
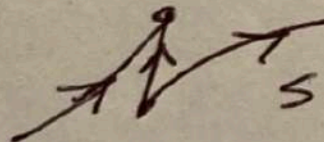
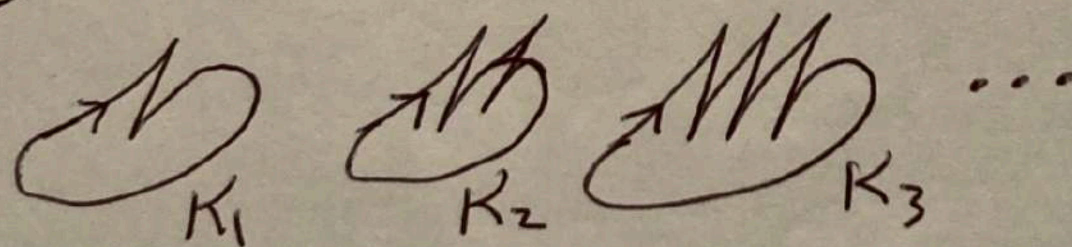
(in 3 MV theory)

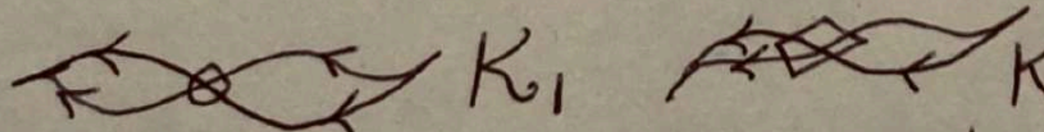


Well of course it is not trivial, but we need a proof. There is a big structure here, largely unexplored.

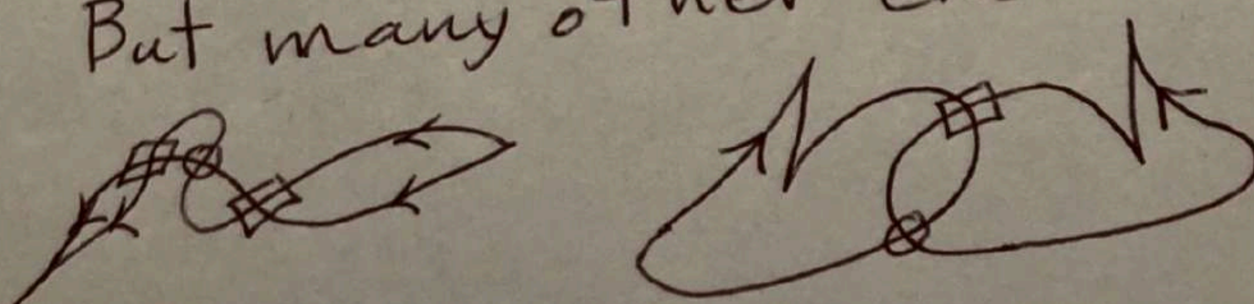
Arrow Polynomial Generalization



- Need  for invariance.
-  survives.
- 



But many other end states:



Now there is a big structure
of end-states such as

and there need some
graphical classification
to sort out the new
doubled virtual arrow
polynomial.

There are many questions
and this just the beginning
of this development.

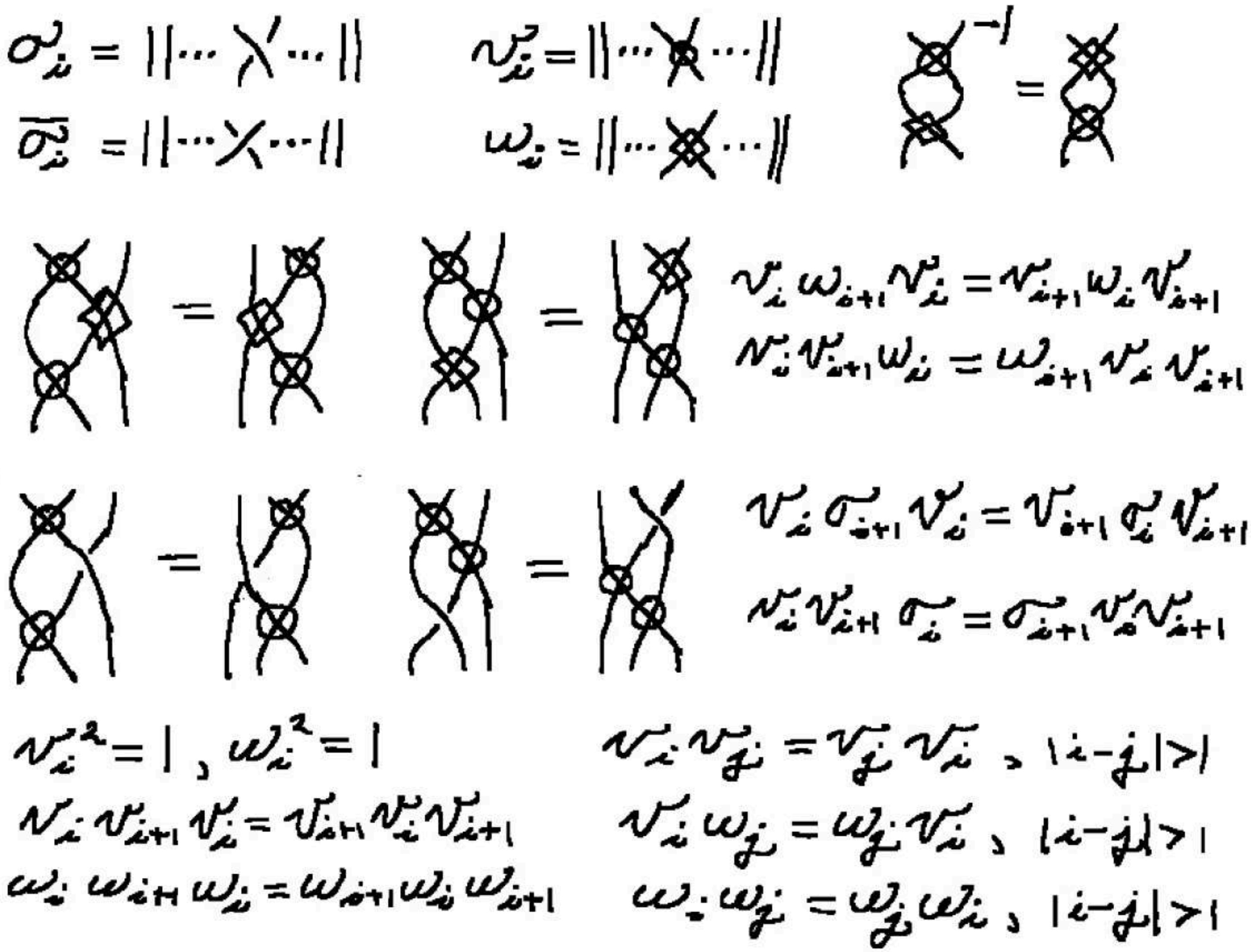


Figure 78: **Multiple Virtual Braid Group**

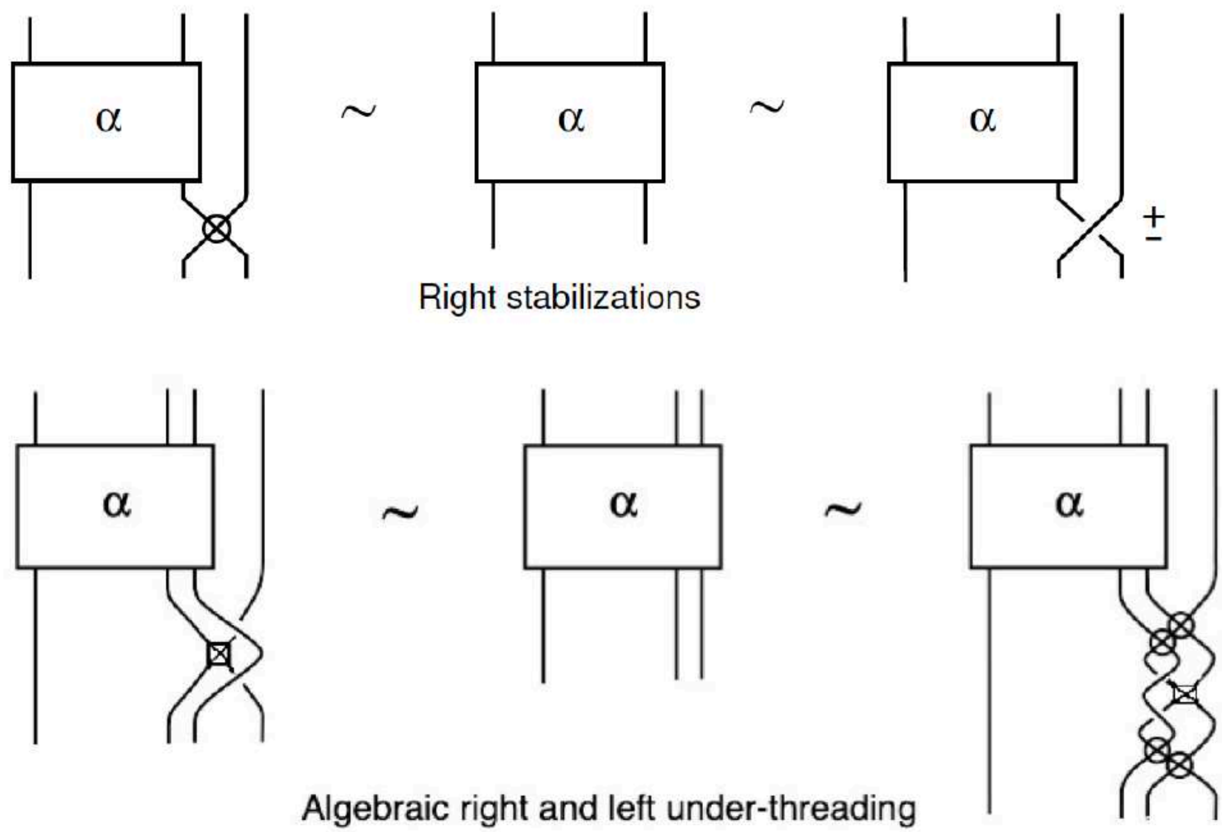


Figure 79: **The Moves (ii), (iii) and (iv) of the Algebraic Markov Theorem.**

Theorem. (Algebraic Markov Theorem for multi-virtuals). Two oriented multi-virtual links are isotopic if and only if any two corresponding virtual braids differ by a finite sequence of braid relations in MVB_∞ and the following moves or their inverses. In the statement below and in Figure 79, v_n stands for any given virtual crossing type.

(i) Virtual and real conjugation: $v_i \alpha v_i \sim \alpha \sim {\sigma_i}^{-1} \alpha \sigma_i$

(ii) Right virtual and real stabilization: $\alpha v_n \sim \alpha \sim \alpha \sigma_n^{\pm 1}$

(iii) Algebraic right under–threading: $\alpha \sim \alpha \sigma_n^{-1} v_{n-1} \sigma_n^{+1}$

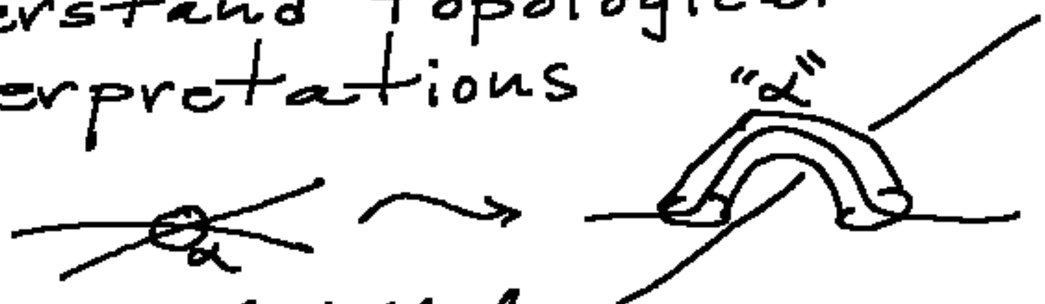
(iv) Algebraic left under–threading: $\alpha \sim \alpha v_n v_{n-1} \sigma_{n-1}^{+1} (v_n)' \sigma_{n-1}^{-1} v_{n-1} v_n,$

where $\alpha, v_i, \sigma_i \in VB_n$ and $v_n, \sigma_n \in VB_{n+1}$ (see Figure 79) and $(v_n)'$ denotes a possibly different virtual crossing type from v_n . Note that in Figure 79 this possible difference in virtual crossing type is indicated by a box at the crossing rather than a circle.

(This result will be in a paper in preparation by LK and S. Lambropoulou.)

Many Problems

- articulate invariants
- relations with graph theory
- understand topological interpretations

e.g. 

labelled handles?

- use for understanding classical knots and knotoids.
- and ...