Colorings, Penrose Evaluations and MultiVirtual Knots & Links

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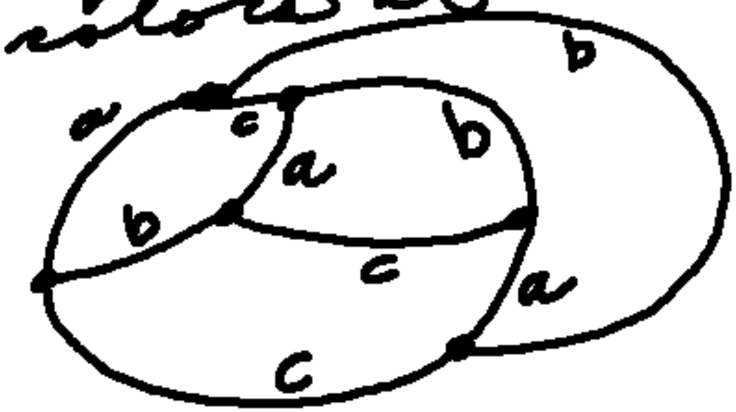
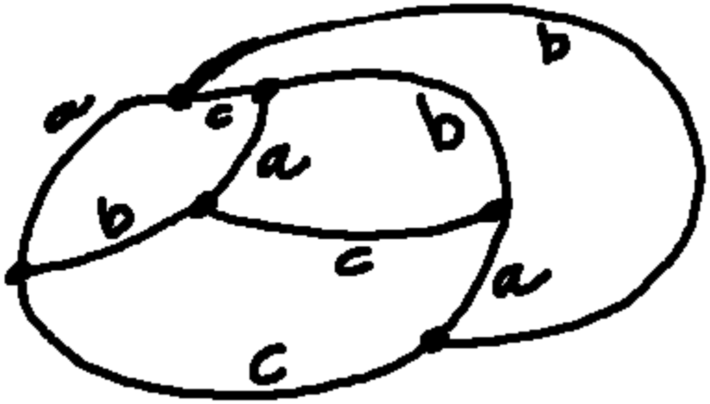
Recall problem of 3-coloring the edges of a cubic graph.

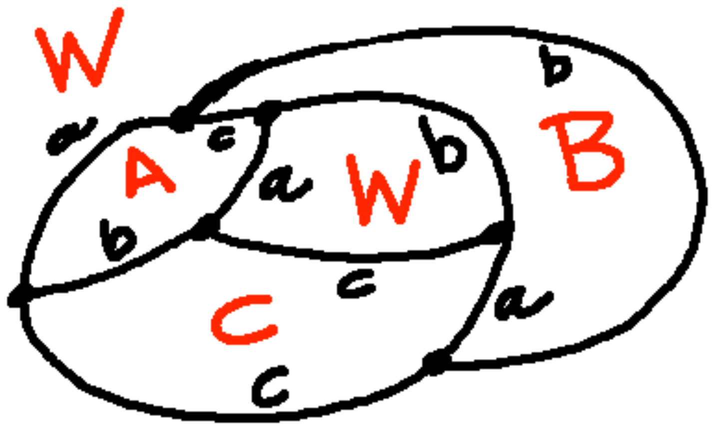
3 colors $\{a, b, c\}$

- all distinct
- require 3 distinct colors at each node.

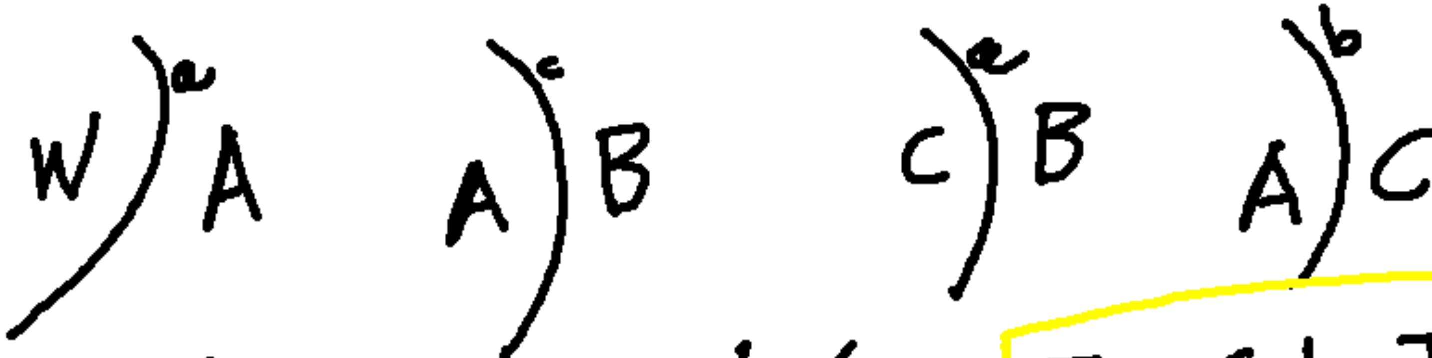
e.g.





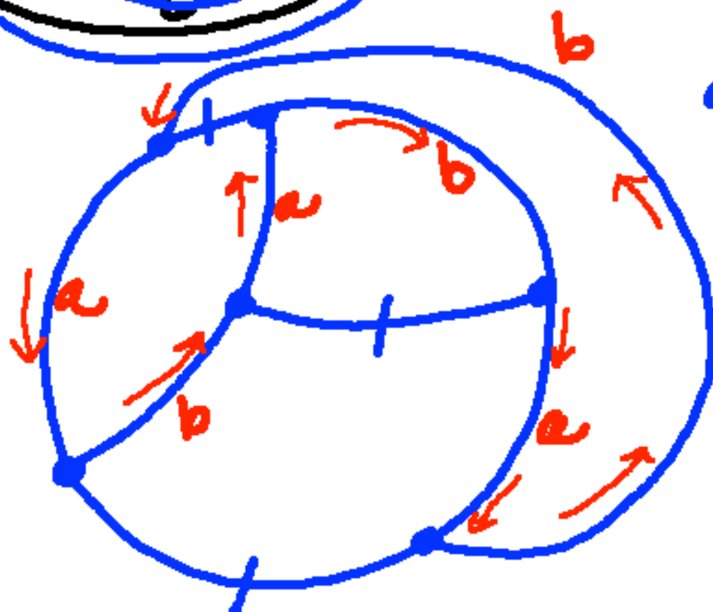


$$G = \{W, A, B, C \mid W = id,\ AB = BA = C,\ A^2 = B^2 = C^2 = W\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$



Four Color Thm
$\Longleftrightarrow$
cubic 1-connected planar are edge 3-colorable

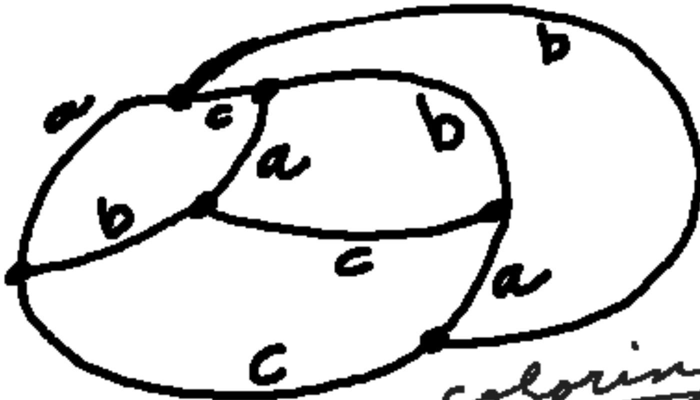
Choose one color (say c) and mark all c edges.



The result is an even perfect matching for the graph G. (Every node taken by the selected edges. Selected edges are disjoint.)

Even PM: Every cycle in (G − PM edges) is an even cycle.

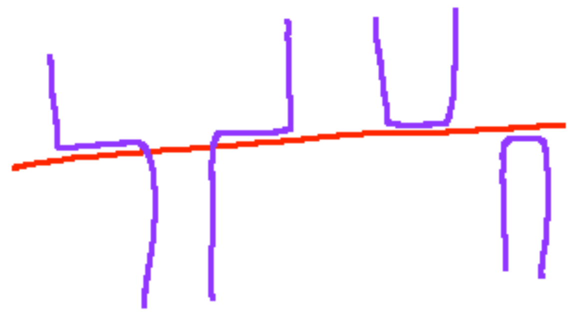
Nota Bene: G is 3 colorable $\Leftrightarrow$ G has an even PM.



coloring

formation

Let a = red
b = blue
c = purple
||
red/blue

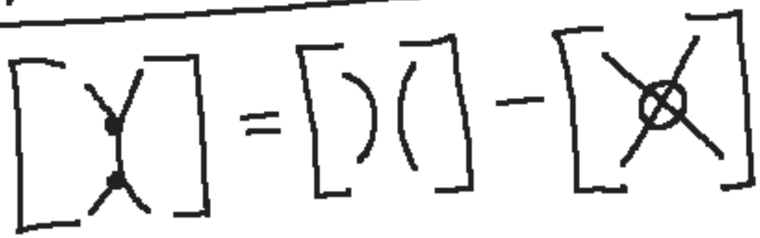


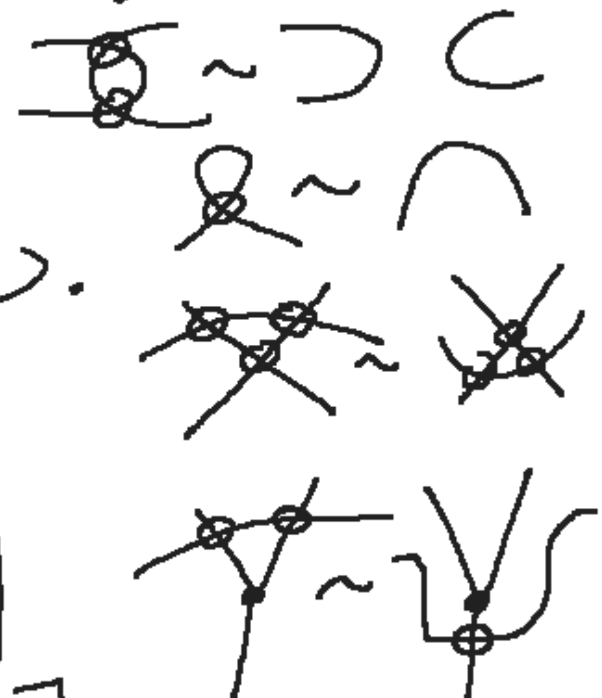
How red meets blue.

One can directly construct infinitely many formations. 4CT ⇒ formations include all planar 3-conn cubic graphs.

The Penrose Formula

 is a virtual crossing



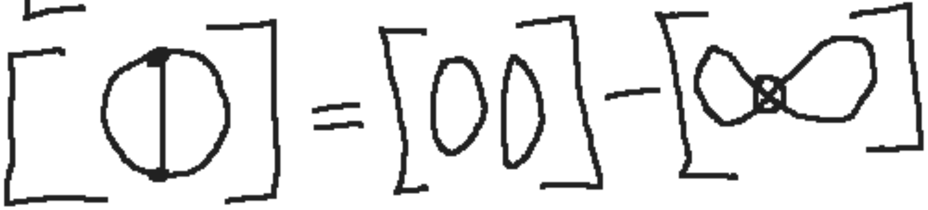


 $= 3$

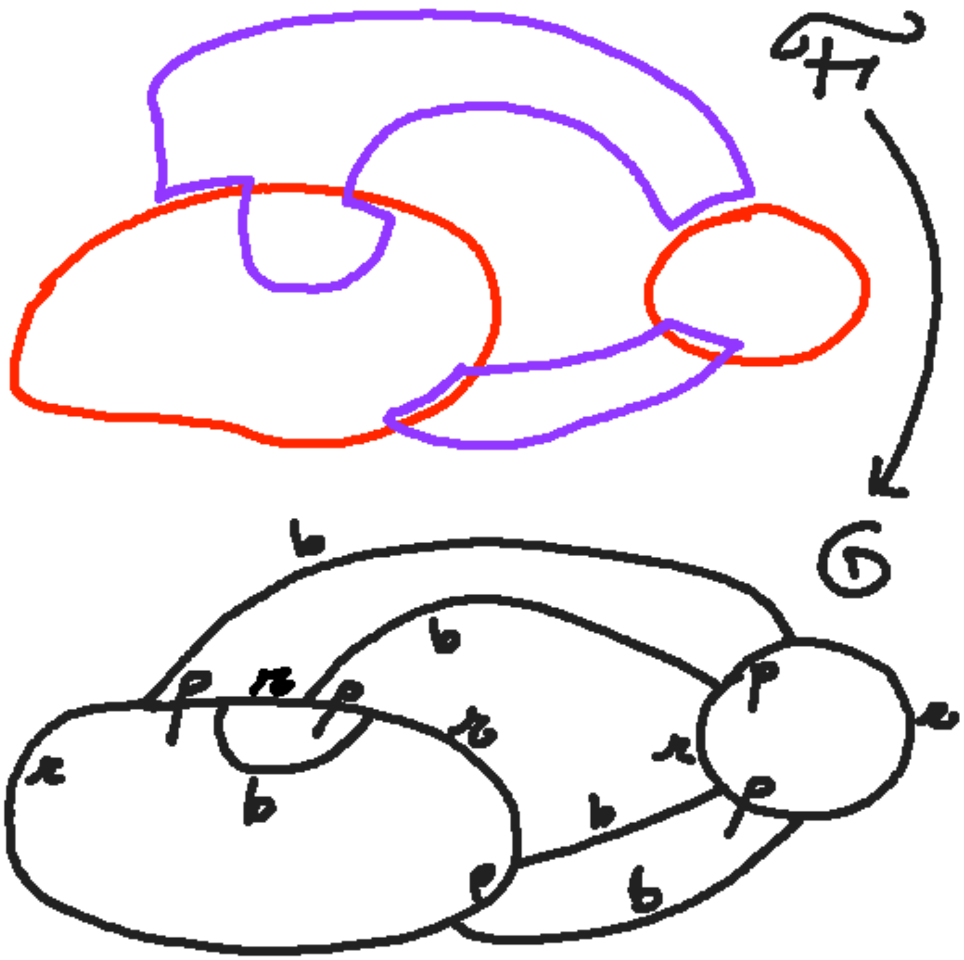
Compute recursively.

Penrose Theorem. G cubic plane graph $\Rightarrow$ $[G] =$ # of 3 colorings of G.

e.g.  $= \phi$.

 $= 3^2 - 3 = 6$.

Formation



Four Color Theorem

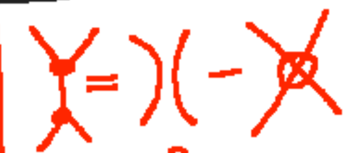
|||

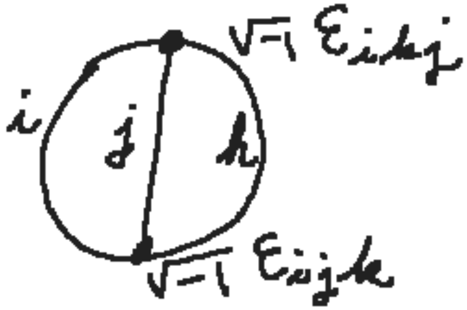
Every 1-connected planar G can be formatted

Thus 4CT $\Longleftrightarrow$ $[G] \neq 0$ whenever G is plane cubic, 1conn.

Penrose Definition (The Ubiquitous Epsilon)

Convert G to a tensor net & contract.





$\longleftrightarrow \sqrt{-1}\,\varepsilon_{ijk}$ where $i,j,k \in \{1,2,3\}$

& $\varepsilon_{ijk} = \begin{cases} \mathrm{sgn}(ijk) & \text{if all distinct.} \\ 0 & \text{if not a permutation of 123.} \end{cases}$

$$[G] = \text{Contraction of Tensor Net}(G)$$

$$= \sum_{\sigma \in \text{colors of } G} \ \prod_{\text{nodes of } G} (\pm\sqrt{-1}) = \langle \sigma \rangle$$

<u>Show</u>: $\langle \sigma \rangle = +1$ for each coloring σ.

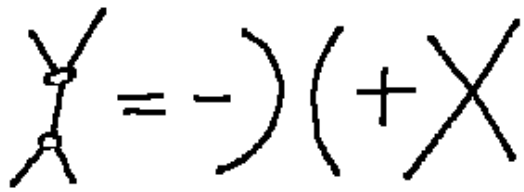
Algebraic Remark

$(a \times b)_k = \sum_{ij} \varepsilon_{ijk}\, a_i b_j$

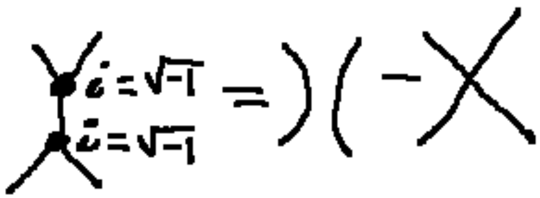

$= \varepsilon_{ijk}$

$a \times b = a \quad b$ Vector cross product


$= \sqrt{-1}\, \varepsilon_{ijk}$



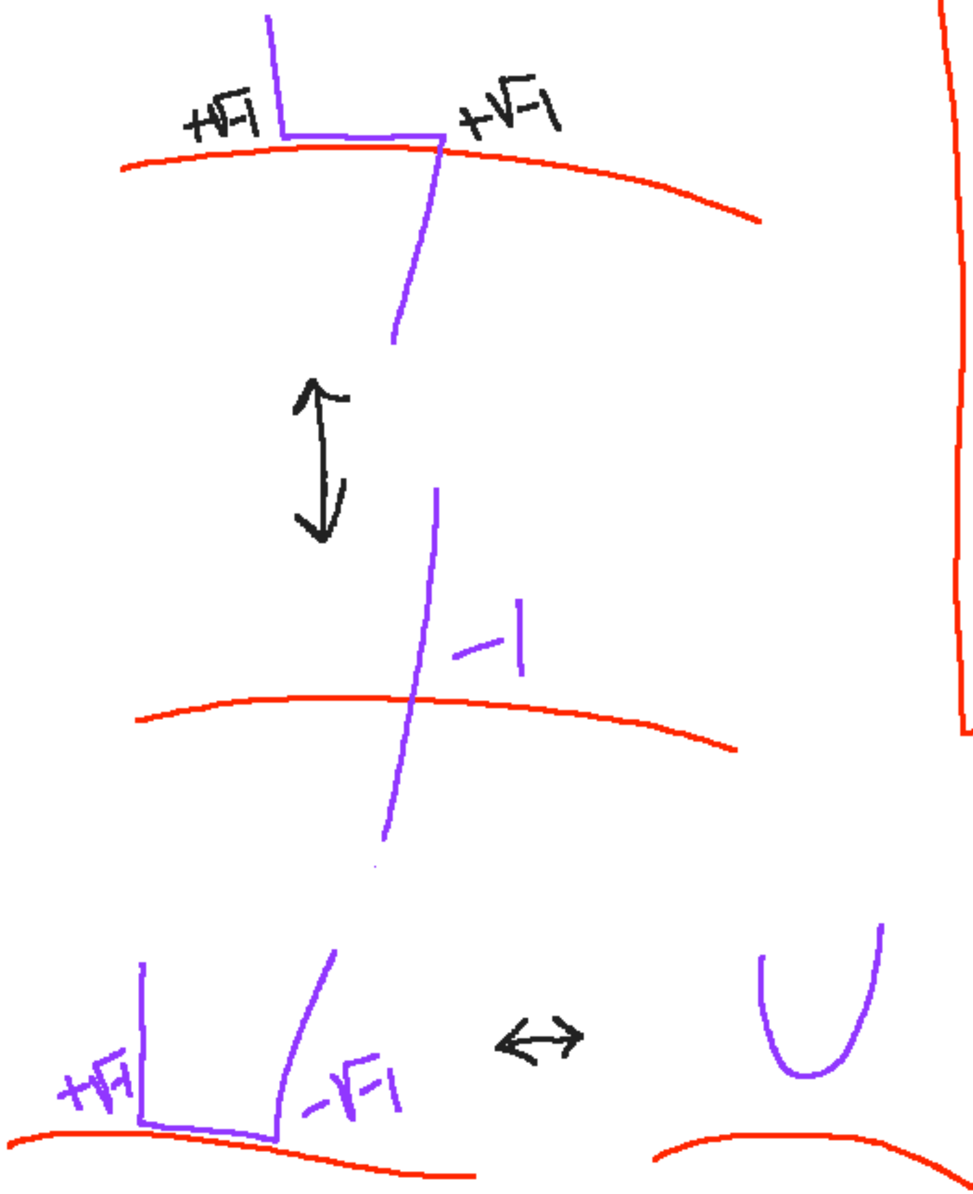
epsilon identity

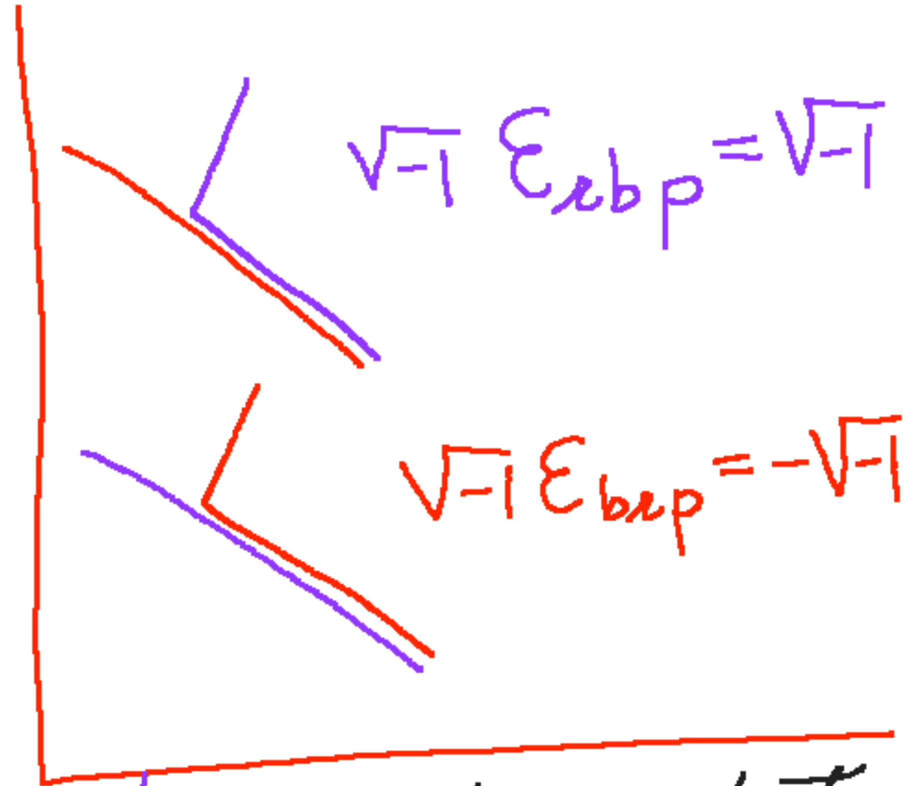


$\leftrightarrow$ Penrose Identity

4CT $\longleftrightarrow$ solvability of equations $(\neq 0)$ of type $(a \times b) \times (c \times d) = (a \times (b \times c)) \times d$ over vector cross product algebra $\{I, J, K\}$.

$\longleftrightarrow$ non-zero products in quaternions.

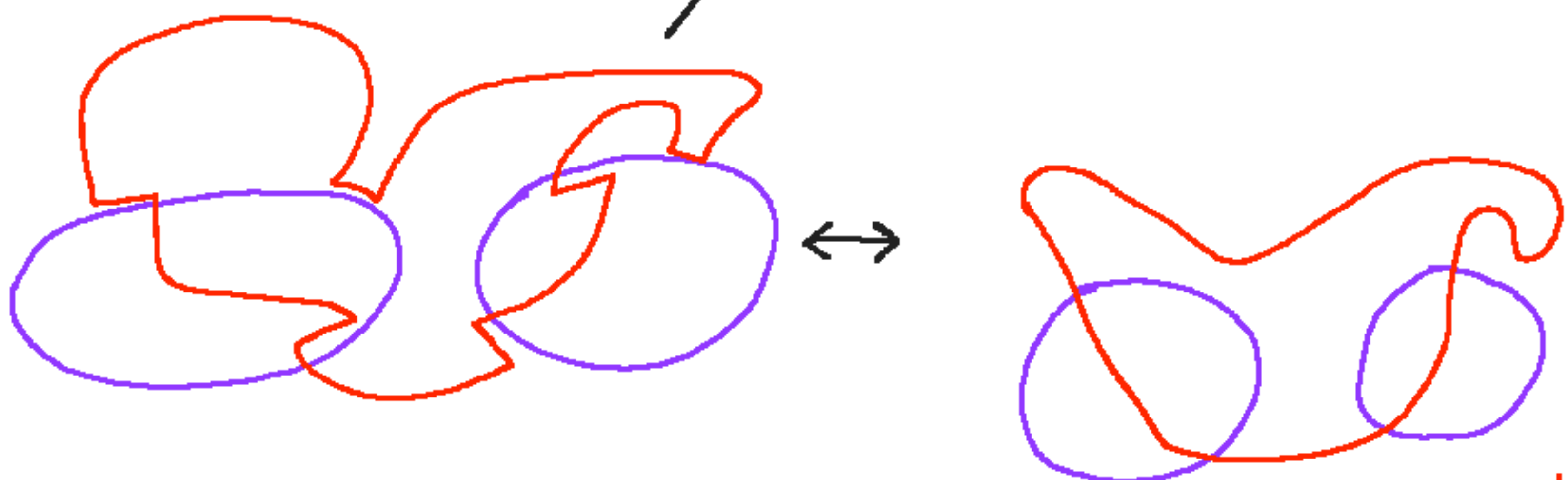




Thus the product of $\pm\sqrt{-1}$'s for a coloring $= (-1)^n$ where $n = \#$ crossings $=$ of "state curves".

But the number of crossings of red + blue curves is <u>even</u> due to the Jordan curve theorem.

$\therefore \; [G] = \#$ of colorings of G when G is a plane cubic graph.



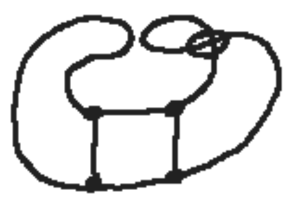
$n = 4 \Rightarrow (-1)^n = +1.$

The Penrose formula does not work for nonplanar graphs (but we shall fix it).

ex.

$G = K_{33}$

Exercise. # of 3 colorings of G is 12.

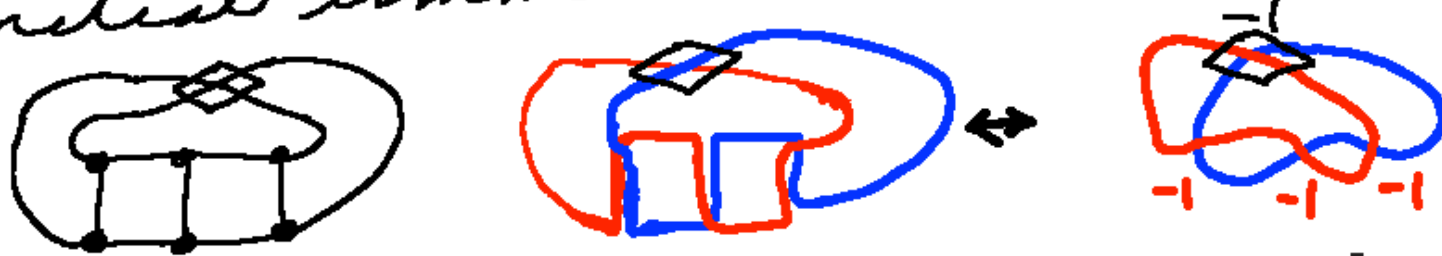
However $[G] =$  $-$  $\simeq$ 

$=$  $-$  $= \emptyset$.

Thus Penrose formula gives zero but we would like it to give 12!

The Fix

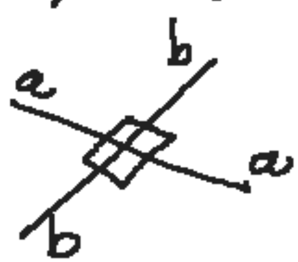
Add a new tensor at the initial immersion crossings.



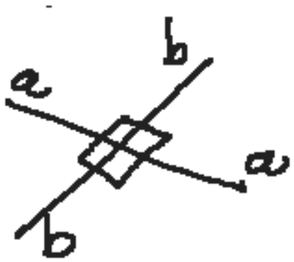
$(-1)^4 = 1$

We change this to a new tensor and a new virtual marker:  "innocuous"

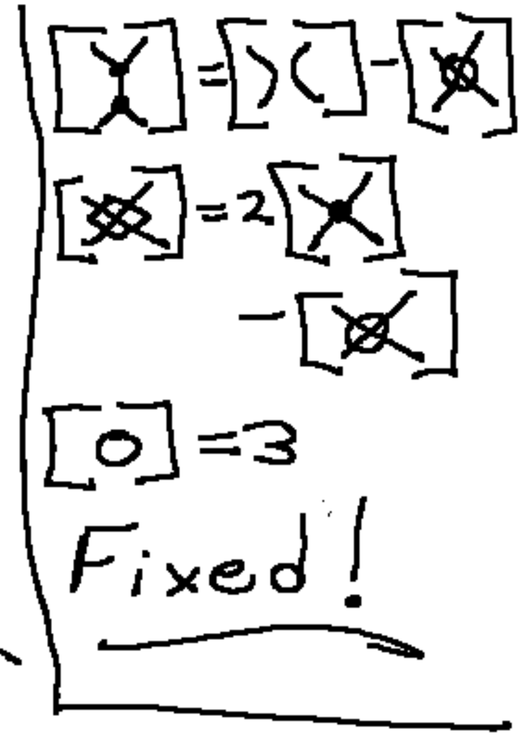
 $= -1$

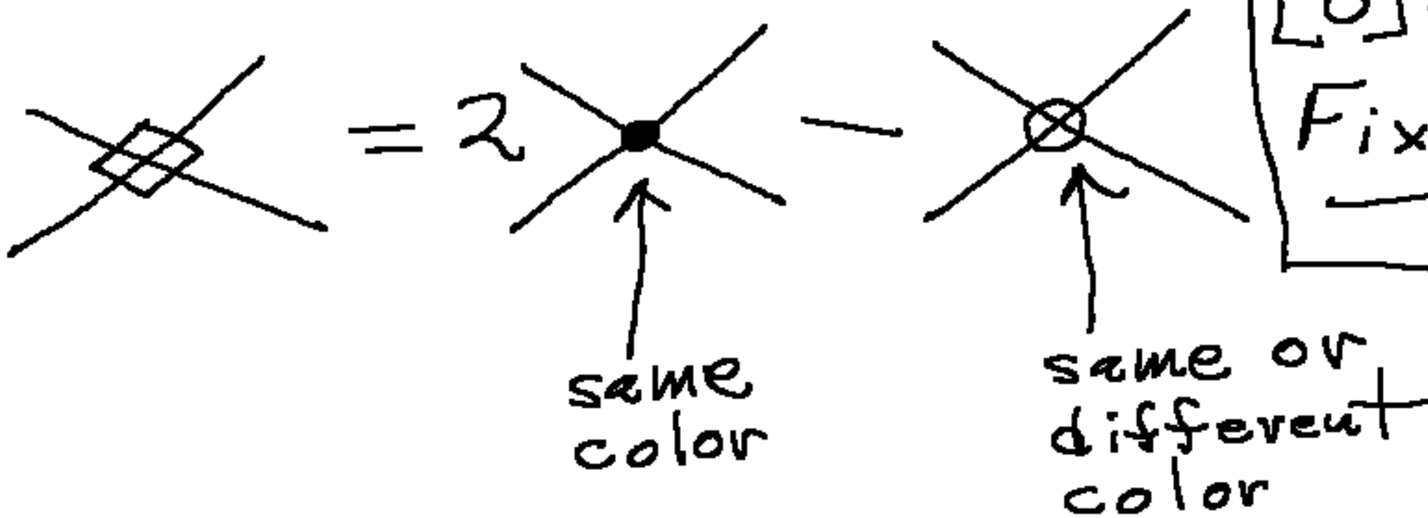
but 

$= \begin{cases} 1 & \text{if } a = b \\ -1 & \text{if } a \neq b \end{cases}$

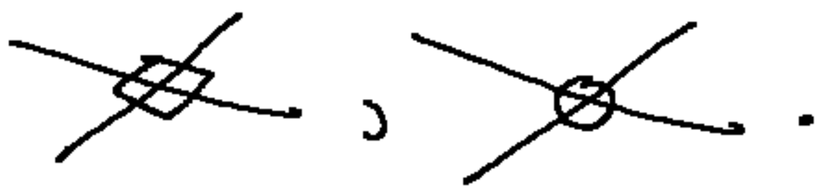


$$= \begin{cases} 1 & \text{if } a=b \\ -1 & \text{if } a\neq b \end{cases}$$

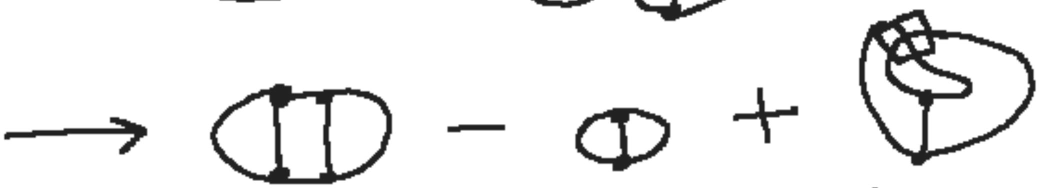


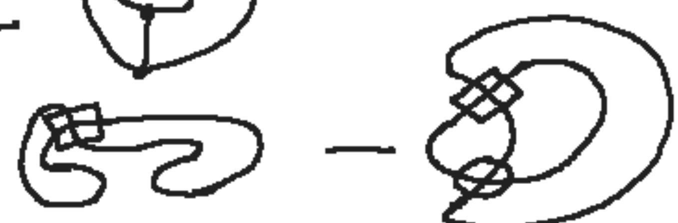


We now have two virtual crossings



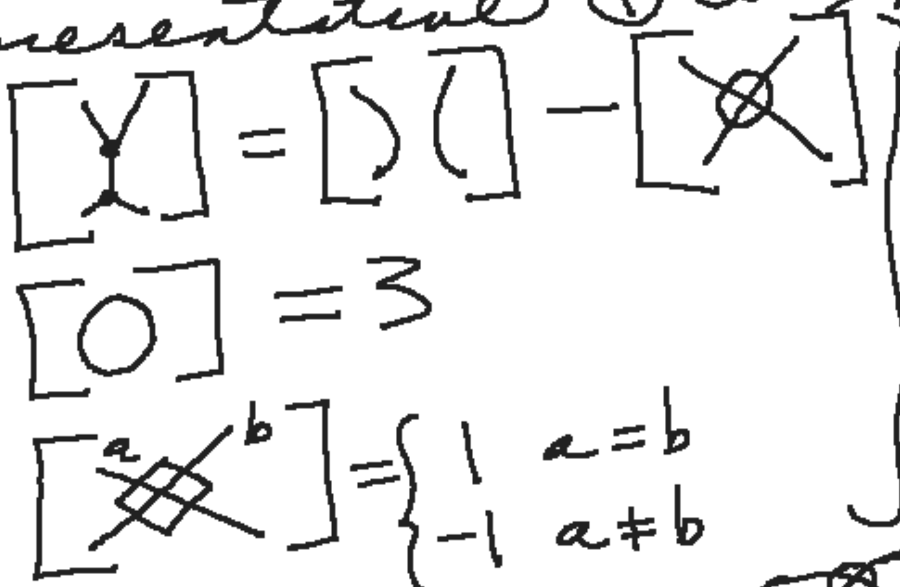
Nota Bene: 



$= 2\times 2\times 3 - 6 +$ 

$= 6+3-[3-6] = 12$

Thus we can now formulate a general Penrose evaluation to recount the number of colorings of arbitrary cubic graphs: Use an immersed representative $G \xrightarrow{f} \mathbb{R}^2$.



In context of a doubled virtual crossing context.

Note that structures like  have a separate chromatic computation.

Onward

1. Generalized Penrose Polynomials for graphs with a perfect matching

2. Generalized doubled virtual knot theory.

Relations between 1) & 2)

Generalize Penrose evaluation to a polynomial.

Try  but $[\bigcirc] = \delta$.

Then evaluation depends upon choice of perfect matching.

So let G be given a perfect matching and define an expansion via

any = same + diff

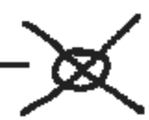
 = same − diff

= 2 (same) − any

= 2 

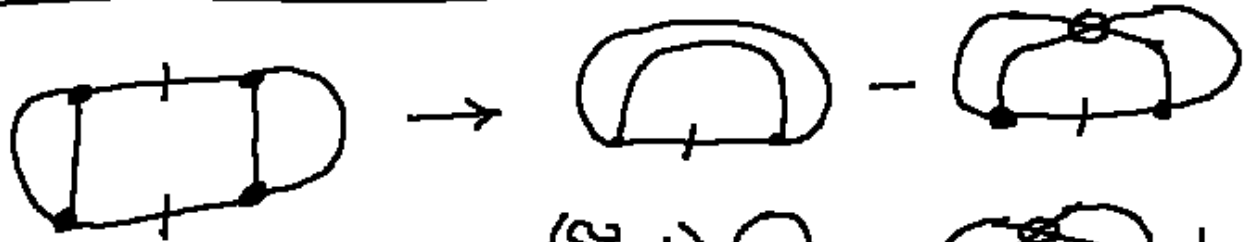
and we need to explain handling  : we want $\delta - \delta(\delta - 1) = 2\delta - \delta^2$

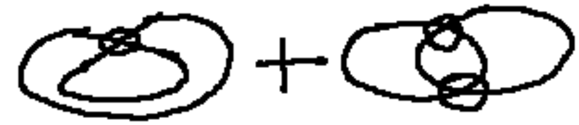
Let  mean "same" so that

 $= \delta$. Let  $= 2$  $-$ 

e.g.  $= 2$  $-$  $= 2\delta - \delta^2$.

 $= (\delta - 1)$ 



$= (\delta - 1) \bigcirc -$ 

$= (\delta - 1)\delta - \delta + \delta^2$

$= 2\delta^2 - 2\delta$

Perfect Matching Polynomials

 $\longrightarrow (\delta - 1)^2 \bigcirc = (\delta - 1)^2 \delta$

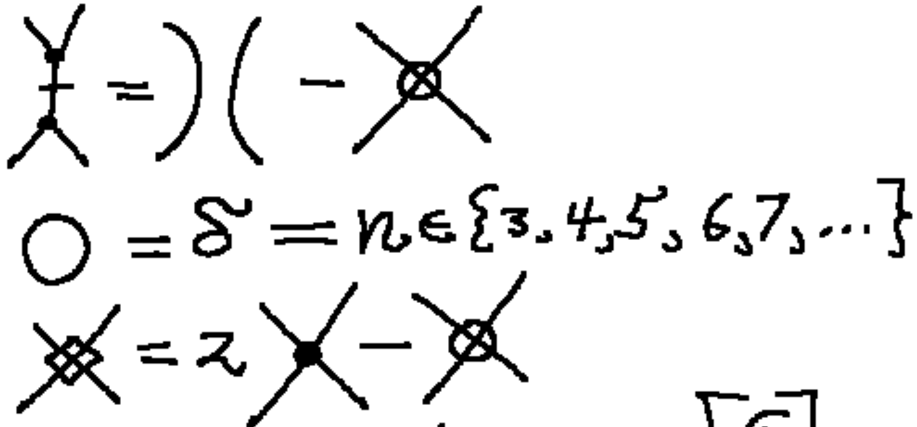
$= \delta^3 - 2\delta^2 + \delta$

N.B. $2 \cdot 3^2 - 2 \cdot 3 = 18 - 6 = 12$

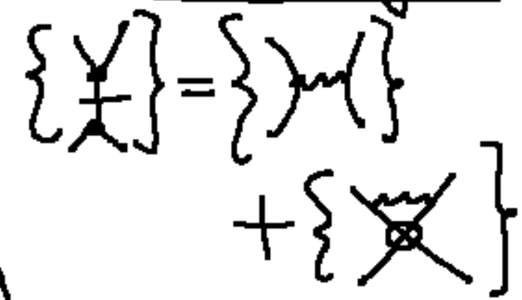
$27 - 18 + 3 = 12$

Agreement at $\delta = 3$.

Lets work with



Tautology



where

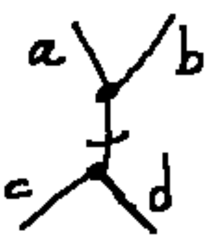
$a)m(b \, : \, a \neq b$

and call the poly in n, $[G]$.

(See paper Scott Baldridge, LK, Ben McCarty)

We relate $[G]$ to a homology theory and we interpret $[G]$ as a coloring count.

We discuss here the counting.



$a \neq b$

$c \neq d$

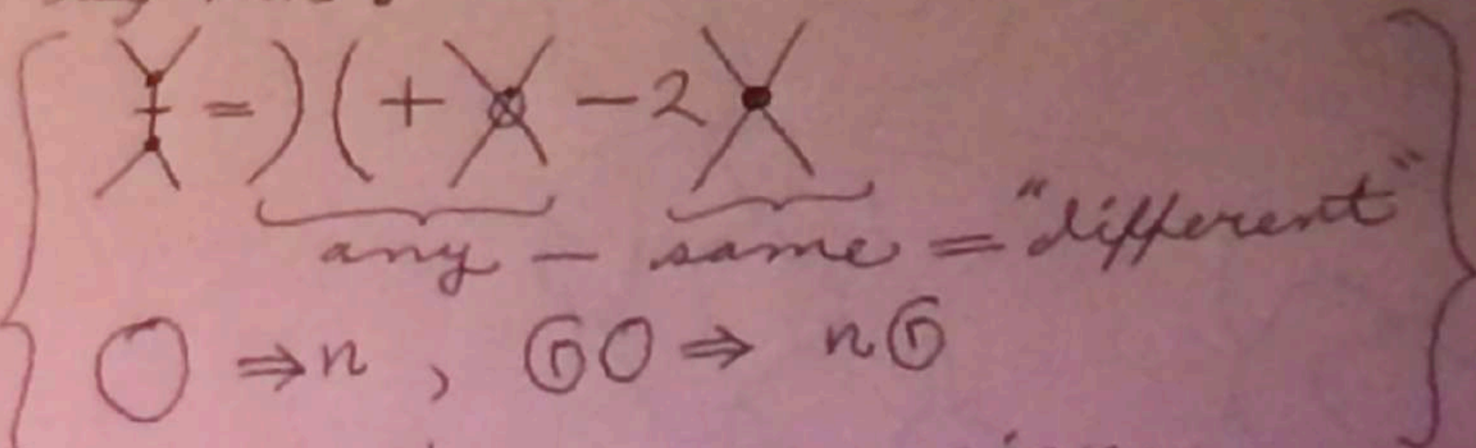
$\{a, b\} = \{c, d\}$

$a, b, c, d \in \{1, 2, \ldots, n\}$

n colors

Color Condition for a given perfect matching).

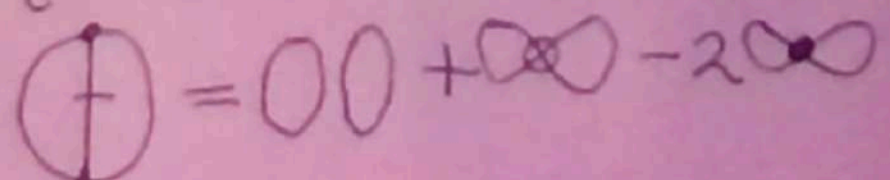
Remark: Thinking chromatically we can say this:



This is a Penrose type expansion and works for all cubic graphs.

$$= \delta^a_c \delta^b_d + \delta^a_d \delta^b_c - 2\eta^{ab}_{cd}$$

ex:



$$= q^2 + q - 2q$$
$$= q^2 - q$$

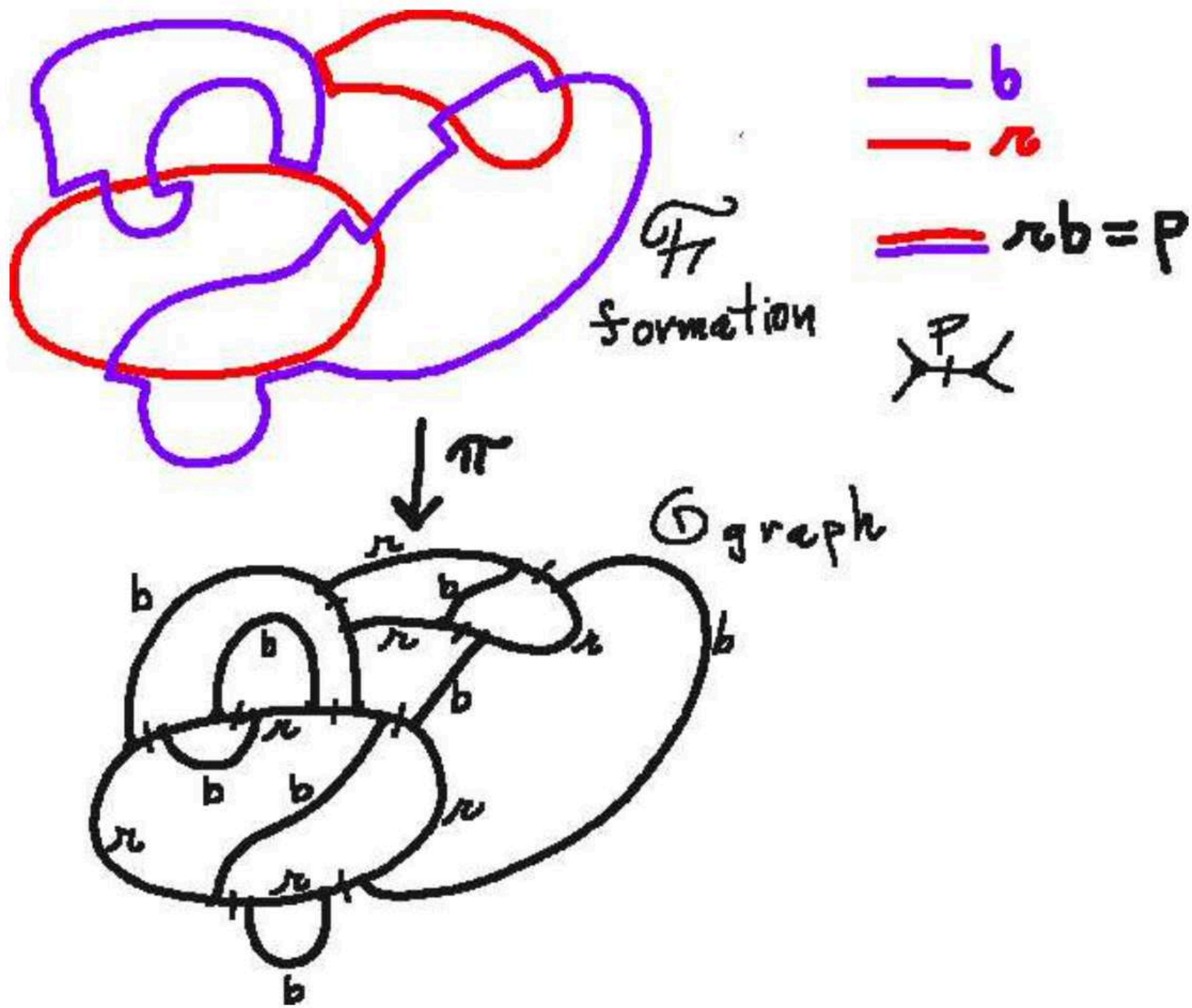


Figure 1: **Standard Formation and Graph**

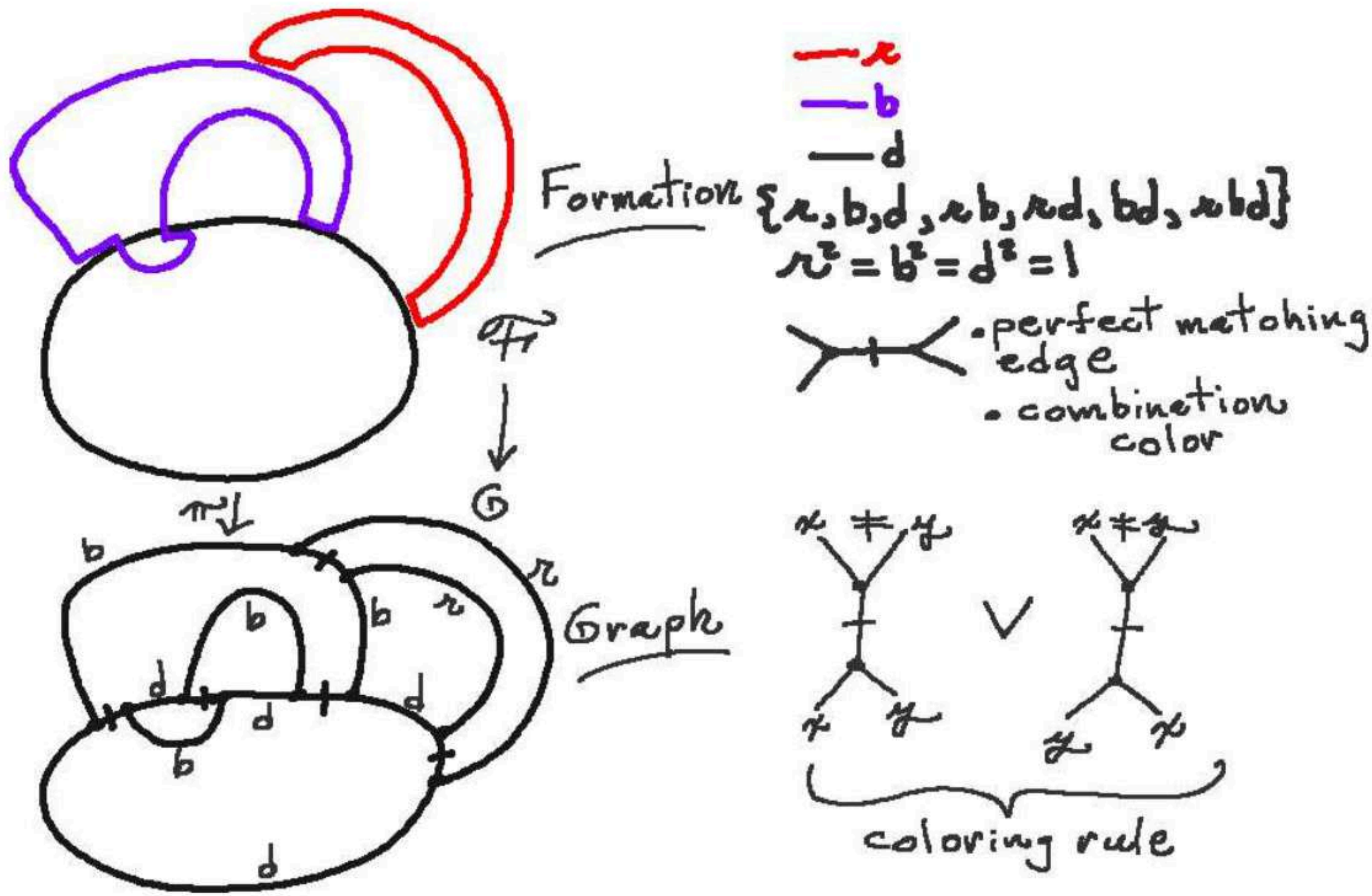
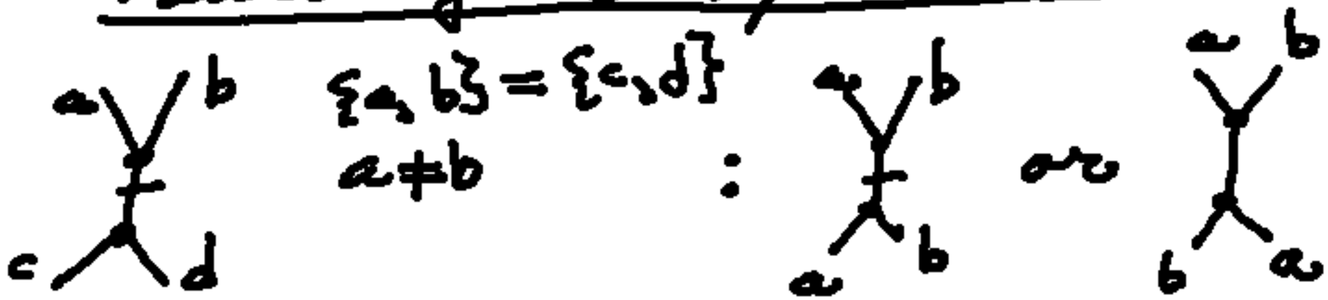
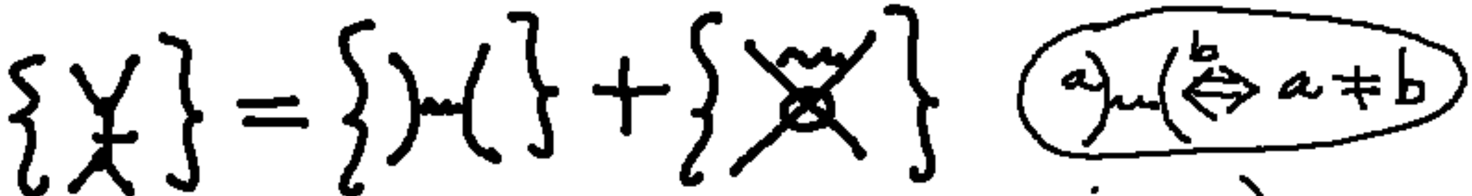


Figure 2: **Generalized Formation and Graph**

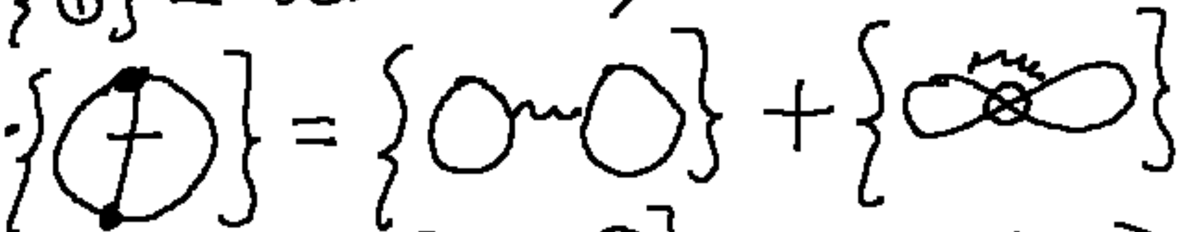
Tautological Expansion





$\{G\}$ = Union of all colorings.

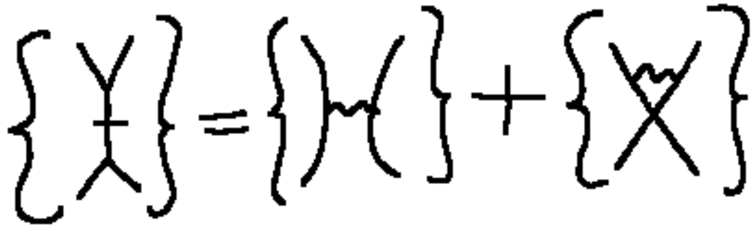
e.g.



$\Rightarrow$ $n(n-1)$ colorings

Compare:

$= n^2 - n$.



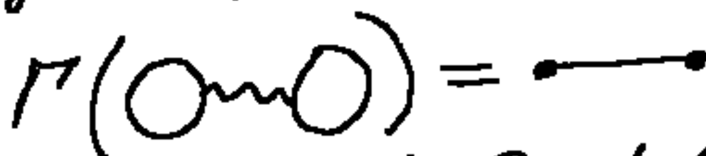
Matching Polynomial

Associate to a state S in this expansion a graph $\Gamma(S)$:

$\text{Loops}(S) = \text{Nodes}(\Gamma(S))$,

$\text{Wiggles}(S) = \text{Edges}(\Gamma(S))$.

e.g.



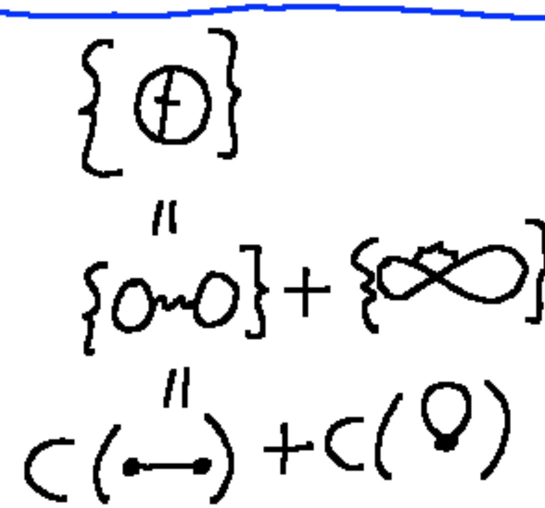
For each state S, define

$\{S\} = C(\Gamma(S))$ = chromatic poly of $\Gamma(S)$ where $C(\bullet) = n = \delta$.

Then $\{G, M\} = \sum_S C(\Gamma(S))$.

cubic graph

perfect matching on G.



$\| $

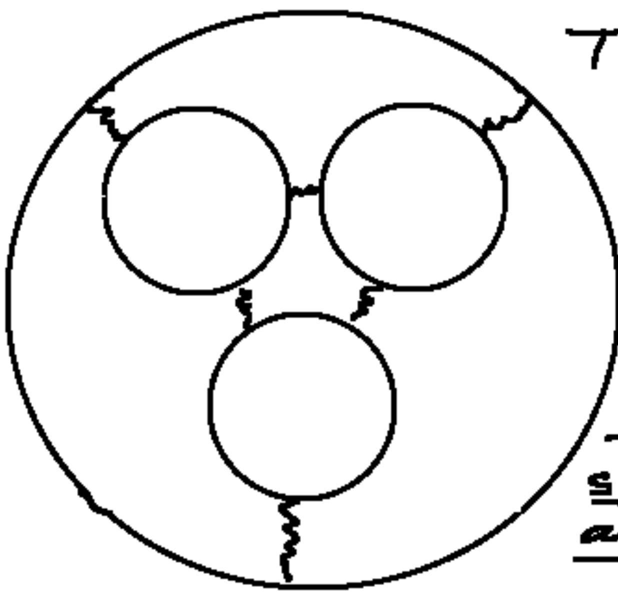
$n(n-1) + \phi$

$\|$

$n(n-1)$

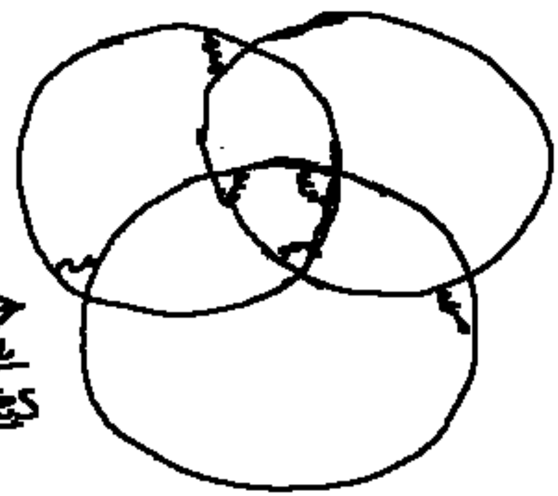
From point of view
of tautological expansion,
start with cycles, local sites,
possibility to switch .

4GT $\Leftrightarrow$ [planer states can be switched to colorable states.]

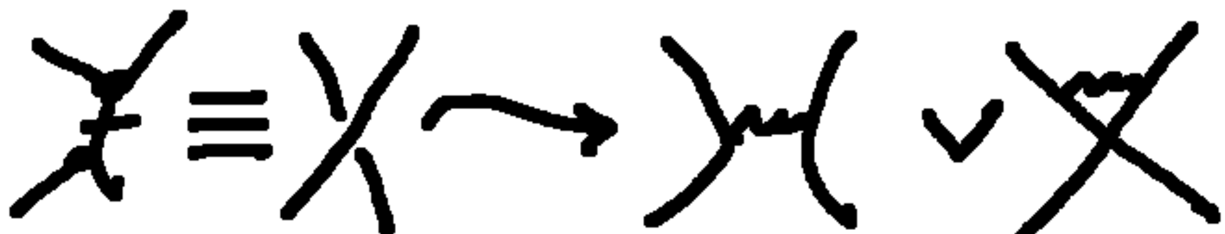


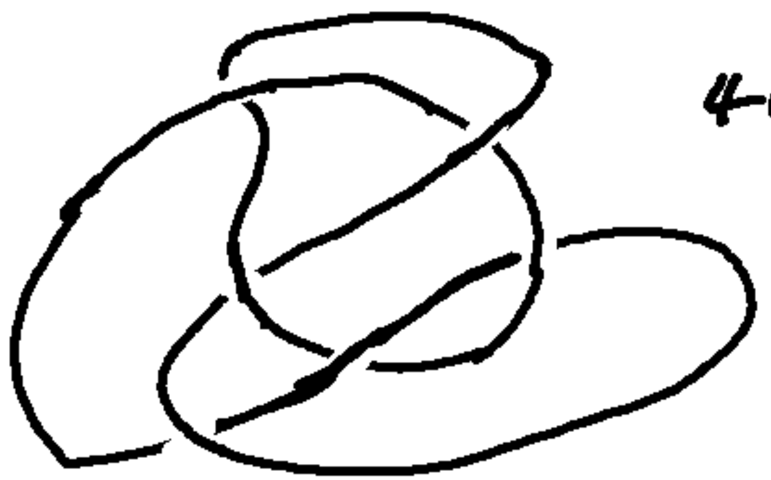
This state is n=3 uncolorable.

switch all sites

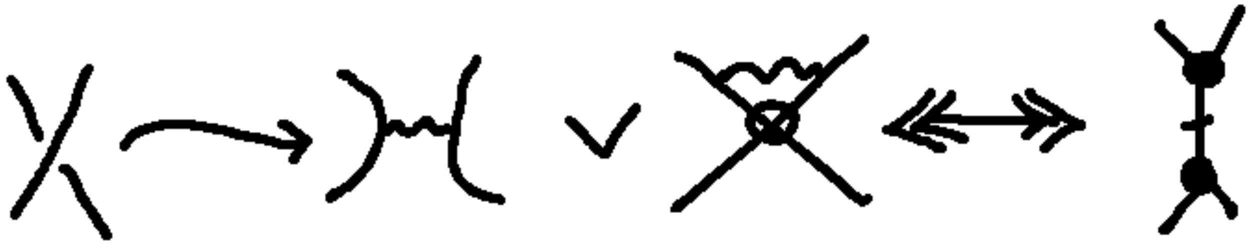


This means you can think in terms of knot/link diagrams.

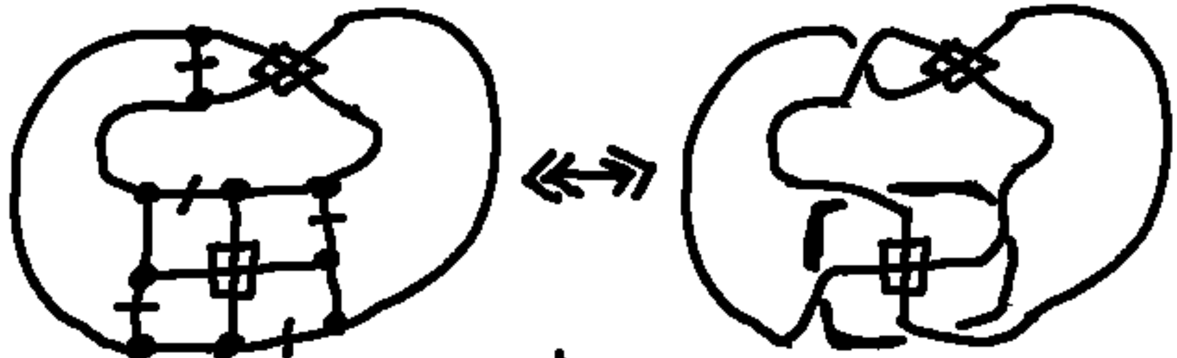




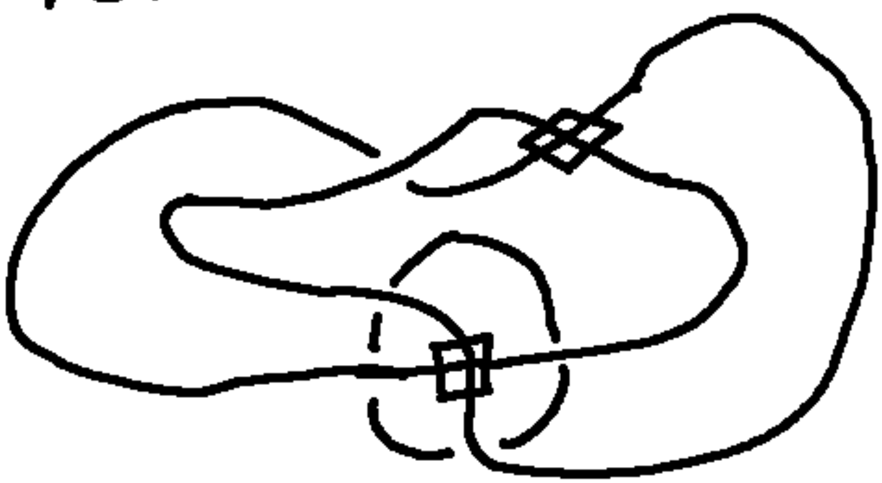
4CT says you can color planar diagrams with 3 colors.



e.g.



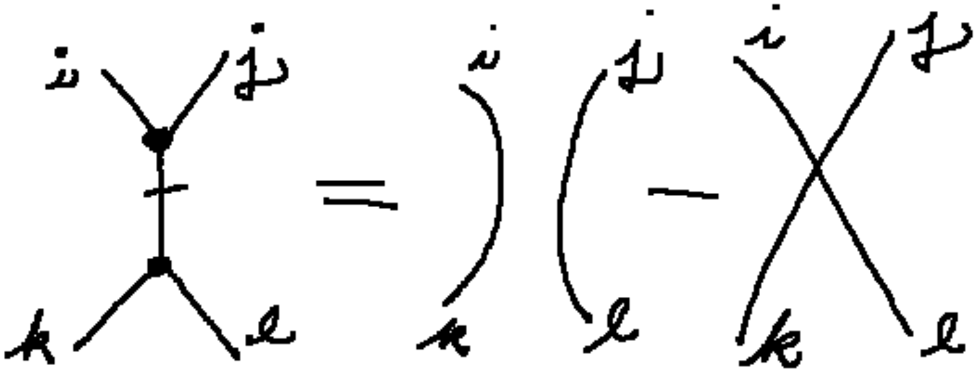
Petersen Graph



(Compare with

.)

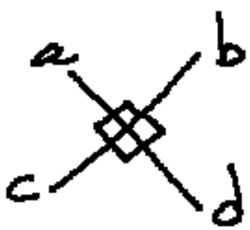
Tensors



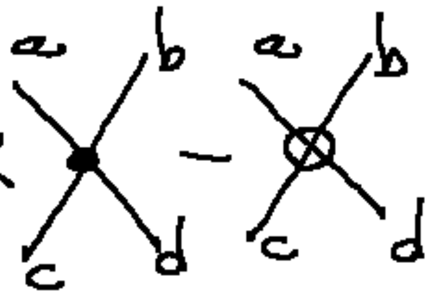
$= \delta^i_j$

Kronecker Delta

 $= n =$ trace of $n \times n$ identity matrix



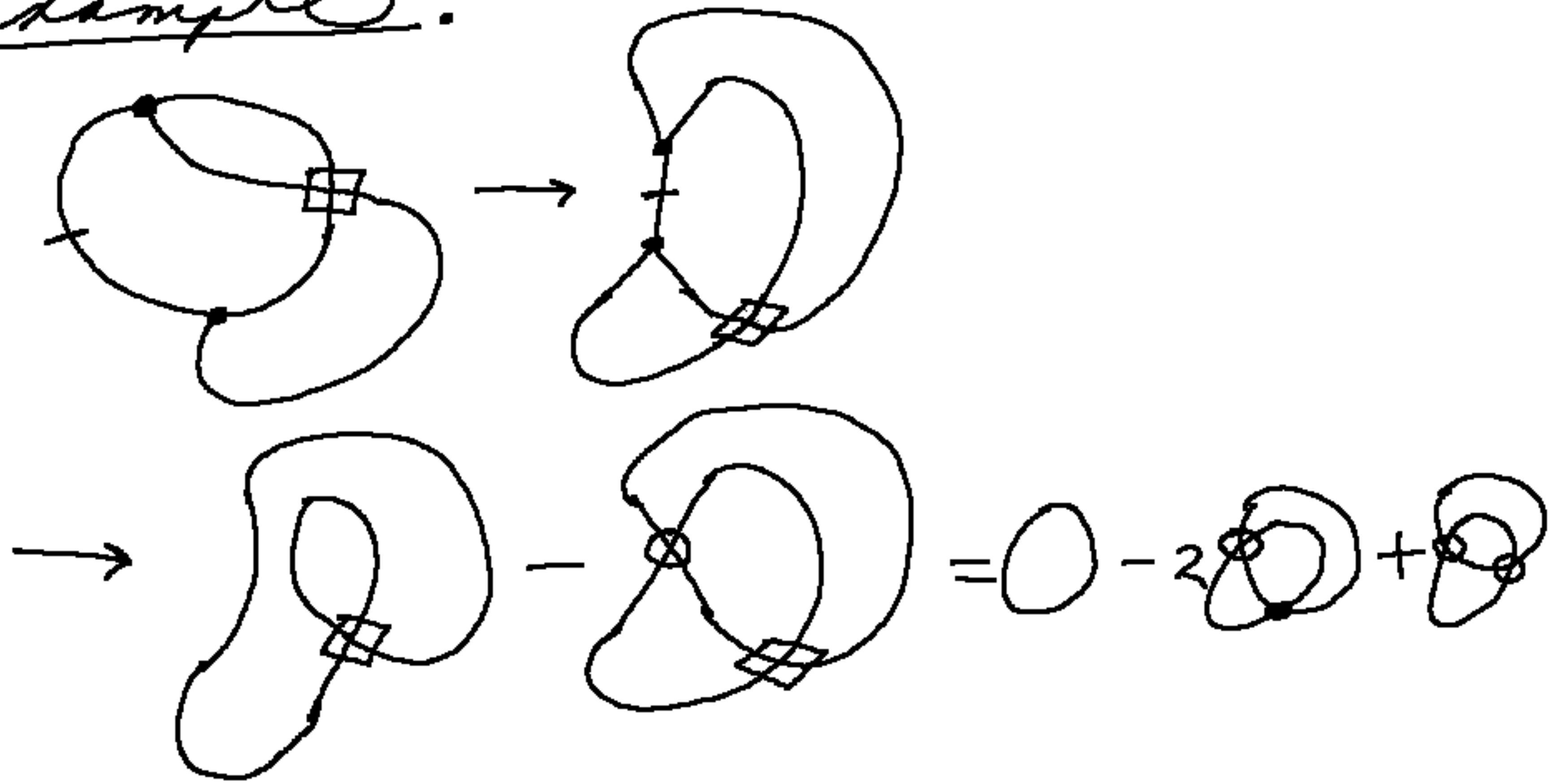
$$= \left\{ \begin{array}{rl} 1 : & a=d=b=c \\ -1 : & a=d \neq b=c \\ 0 : & \text{else} \end{array} \right\} = 2$$



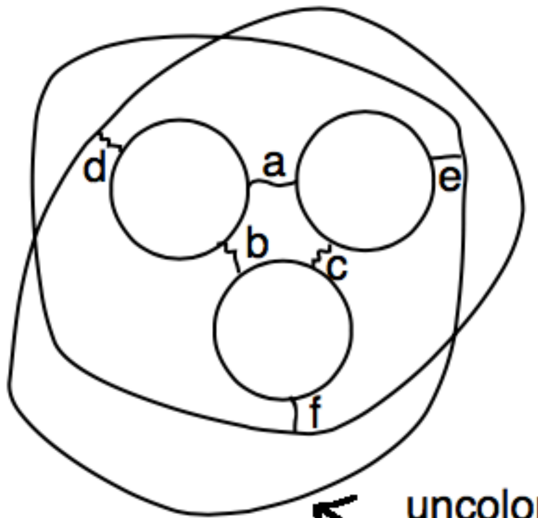
$\Rightarrow$ 

The same arguments as before show that 1) the $\neq 0$ tensor states are all the colorings.
& 2) each contributes $+1$. //

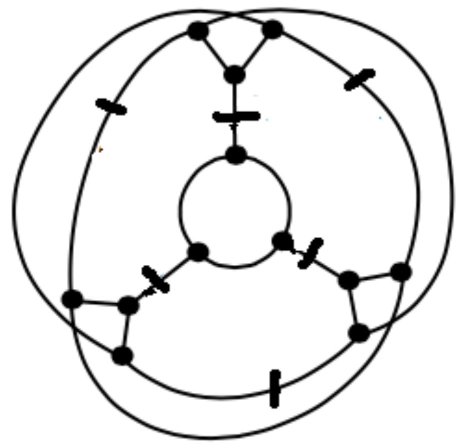
Example.



$$= n - 2n + n^2 = n(n-1).$$



uncolorable

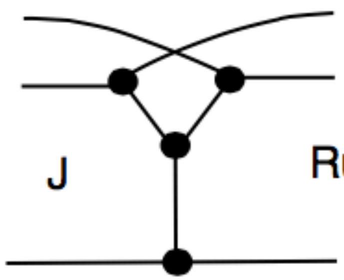


J_3

J_3 is not colorable in 3 colors.

But J_3 can be colored with 4 colors.

(give outer loop a 4th color)



Rufus Isaac's J Construction.

We now examine polynomials for snarks. Here $P(J_3, n)$

$$= n(-6+11n-6n^2+n^3)$$

$$= \begin{cases} 0, & n=3 \\ 24, & n=4 \end{cases}$$

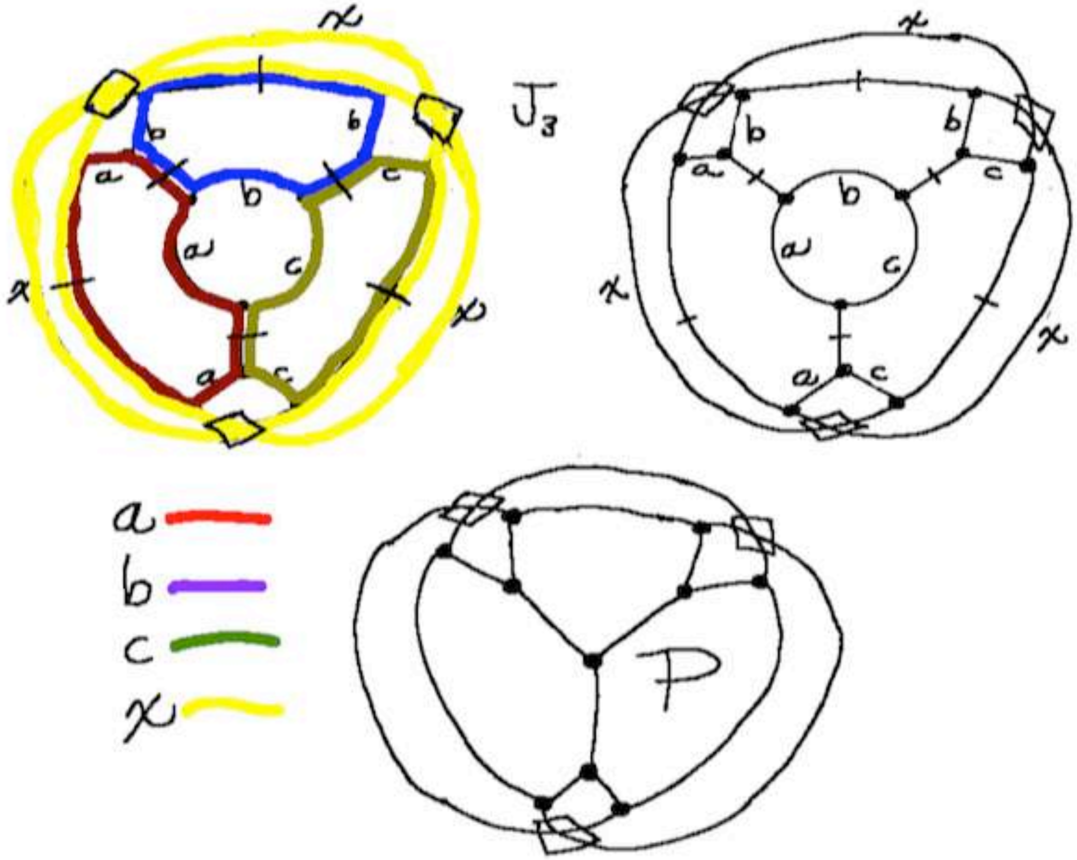
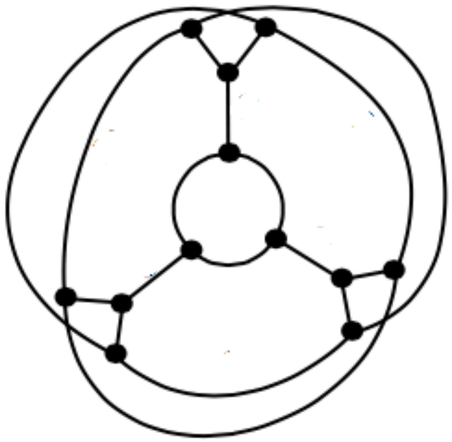
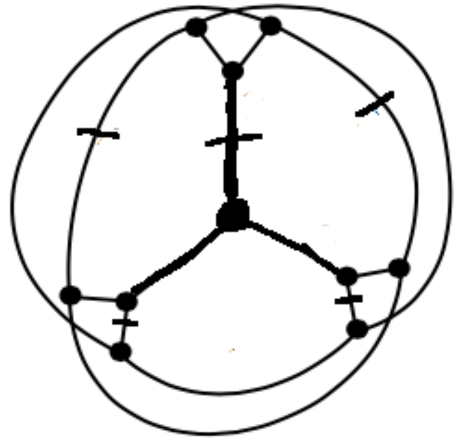


Figure 20: **Isaacs J_3 can be PM-colored with four colors (but not with three colors).**

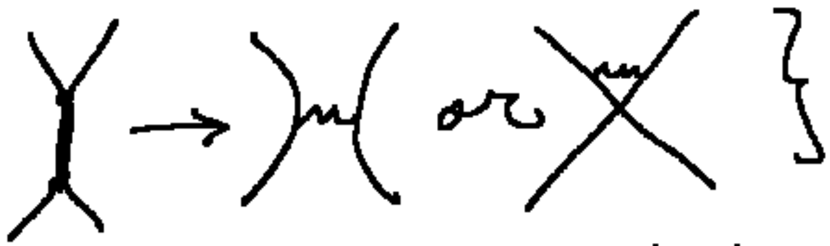


J_3



P

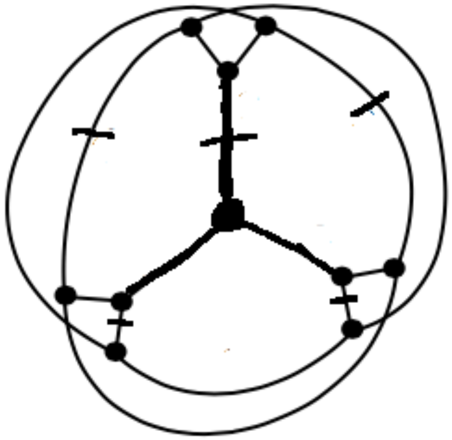
J_3 contracts to the minimal uncolorable Petersen Graph.



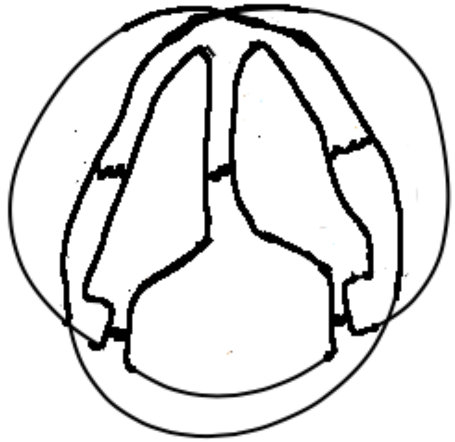
general coloring possibility. using n colors.

__Fact__: P cannot be colored with n colors for any n. Call P __strongly uncolorable__.

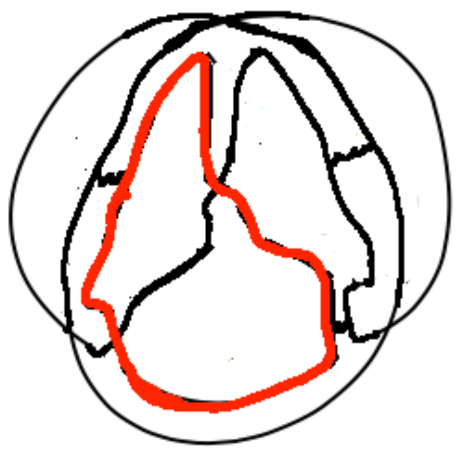
__Conjecture__. If G trivalent is strongly uncolorable, then $G \supset P$ as a __substructure__.

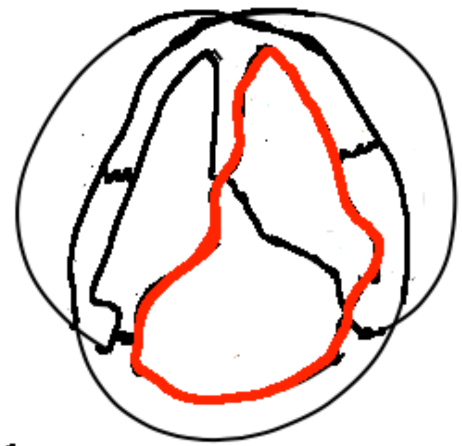


P



not colorable





multiple component states
but still uncolorable