LECTURE NOTES : INTRODUCTION TO DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The aim of this manuscript is to provide a short and accessible introduction to the modern theory of dispersive partial differential equations (PDE). It consists mainly of three parts which are organized as follows:

- Part I focuses on the well–posedness and scattering theory of the semilinear Schrödinger equation.
- Part II concentrates on basic well–posedness theory for the Korteveg–de Vries equation.
- Part III develops the well–posedness theory of dispersive partial differential equations on the half line. We use the cubic nonlinear Schrödinger equation as a prototypical example.

Dislaimer. The notes are prepared as a study tool for the participants of the summer school "Introduction to dispersive PDE". We tried to include many of the relevant references. However it is inevitable that we had to make sacrifices in the choice of the material that is included in the notes. As a consequence, there are many important works that we could not present in the notes.

1. What is a dispersive PDE

Informally speaking, a partial differential equation (PDE) is characterized as dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. As an example consider the linear homogeneous Schrödinger equation on the real line

$$iu_t + u_{xx} = 0, \tag{1.1}$$

for a complex valued function $u = u(x,t)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}$. If we try to find a solution in the form of a simple wave

$$u(x,t) = Ae^{i(kx-\omega t)},$$

we see that it satisfies the equation if and only if

$$\omega = k^2. \tag{1.2}$$

The relation (1.2) is called the dispersive relation corresponding to the equation (1.1). It shows that the frequency is a real valued function of the wave number. If we denote the phase velocity by $v = \frac{\omega}{k}$, we can write the solution as $u(x,t) = Ae^{ik(x-v(k)t)}$ and notice that the wave travels with velocity k. Thus the wave

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propagates in such a way that large wave numbers travel faster than smaller ones[1]. If we add nonlinear effects and study for example

$$iu_t + u_{xx} + |u|^{p-1}u = 0,$$

we will see that even the existence of solutions over small times requires delicate techniques.

Going back to the linear homogenous equation (1.1), let us now consider

$$u_0(x) = \int_{\mathbb{R}} \hat{u}_0(k) e^{ikx} dk.$$

For each fixed k the wave solution becomes

$$u(x,t) = \hat{u}_0(k) e^{ik(x-kt)} = \hat{u}_0(k) e^{ikx} e^{-ik^2 t}.$$

Summing over k (integrating) we obtain the solution to our problem

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(k) e^{ikx - ik^2 t} dk.$$

Since $|\hat{u}(k,t)| = |\hat{u}_0(k)|$ we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Thus the conservation of the L^2 norm (mass conservation or total probability) and the fact that high frequencies travel faster, leads to the conclusion that not only the solution will disperse into separate waves but that its amplitude will decay over time. This is not anymore the case for solutions over compact domains. The dispersion is limited and for the nonlinear dispersive problems we notice a migration from low to high frequencies. This fact is captured by zooming more closely in the Sobolev norm

$$\|u\|_{H^s} = \left(\int |\hat{u}(k)|^2 (1 + |k|)^{2s} dk \right)^{1/2}$$

and observing that it actually grows over time.

Another characterization of dispersive equations comes from the observation that the space-time Fourier transform (we usually denote by (ξ, τ) the dual variables of (x,t)) of their solutions are supported on hyper-surfaces that have non vanishing Gaussian curvature. For example taking the Fourier transform of the solution of the linear homogeneous Schrödinger equation

$$iu_t + \Delta u = 0,$$

for $x \in \mathbb{R}^n$ and $t \geq 0$, we obtain that $u(\xi, \tau)$ is supported[2] on $\tau = |\xi|^2$.

In dispersive equations there is usually a competition between dispersion that over time smooths out the initial data (in terms of extra regularity and/or in terms of extra integrability) and the nonlinearity that can cause concentration, blow-up or even ill-posedness in the Hadamard sense. We focus our attention on the following two dispersive equations:

[1]Trying a wave solution of the same form to the heat equation $u_t - u_{xx} = 0$, we obtain that the ω is complex valued and the wave solution decays exponential in time. On the other hand the transport equation $u_t - u_x = 0$ and the one dimensional wave equation $u_{tt} = u_{xx}$ have traveling waves with constant velocity.

[2]In this light the linear wave equation in dimension higher than two is dispersive as the solution is supported on the cone $\tau = |\xi|$.

- Nonlinear Schrödinger (NLS) equation given by
$$iu_t + \Delta u + f(u) = 0,$$
where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$.

- The Korteweg-de Vries equation (KdV) given by
$$u_t + u_{xxx} + uu_x = 0,$$
where $u : M \times \mathbb{R} \to \mathbb{R}$ with $M \in \{\mathbb{R}, \mathbb{T}\}$

as two prime examples. However the methods that are reviewed in these notes apply equally well to other dispersive PDE. The competition mentioned above comes to light in a variety of ways. On one hand, we have the case of the NLS (2.1) of defocusing type with a polynomial nonlinearity of high enough power. In this case the global energy solutions that we will obtain satisfy additional decay estimates that over time weaken the nonlinear effects. It is then possible to compare the dynamics of the NLS with the linear problem and show that as $t \to \infty$ the nonlinearity "disappears" and the solution approaches the free solution. On the other hand, we have the case of the KdV equation. There, the dispersion and the nonlinearity are balanced in such away that solitary waves (global traveling wave solutions) exist for all times. These traveling waves are smooth solutions that prevent the equation from scattering even on the real line. Many different phenomena intertwine with dispersion but in these notes we can develop and partially answer only the most basic of questions. For more details the reader can consult [9, 11, 25, 55, 70, 71].

To analyze further the properties of dispersive PDE and outline some recent developments we start with a concrete example.

2. The semi-linear Schrödinger equation.

Consider the semi-linear Schrödinger equation (NLS) in arbitrary dimensions
$$\begin{cases} iu_t + \Delta u + \lambda |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad \lambda \pm 1, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{2.1}$$
for any $1 < p < \infty$. Here $H^s(\mathbb{R}^n)$ denotes the s Sobolev space, which is a Banach space that contains all functions that along with their distributional s-derivatives belong to $L^2(\mathbb{R}^n)$. This norm is equivalent (through the basic properties of the Fourier transform) to
$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty.$$
When $\lambda = -1$ the equation is called defocusing and when $\lambda = 1$ it is called focusing.

NLS is a basic dispersive model that appears in nonlinear optics and water wave theory. Before we outline basic properties and questions of interest concerning solutions to (2.1), we review symmetries of the equation.

2.1. **Symmetries of the equation.** One of the questions that we shall consider is the following: for what values of $s \in \mathbb{R}$ one can expect reasonable solutions? The symmetries of the equation (2.1) can be very helpful in addressing this question.

(1) A symmetry that we shall often mention is the **scaling symmetry**, that can be formulated as follows. Let $\lambda > 0$. If u is a solution to (2.1) then

$$u^{\lambda}(x,t) = \lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda},\ \frac{t}{\lambda^2}), \quad u_0^{\lambda} = \lambda^{-\frac{2}{p-1}} u_0(\frac{x}{\lambda}),$$

is a solution to the same equation. If we compute $\|u_0^{\lambda}\|_{\dot{H}^s}$ we see that

$$\|u_0^{\lambda}\|_{\dot{H}^s} = \lambda^{s_c - s} \|u_0\|_{\dot{H}^s}$$

where $s_c = \frac{n}{2} - \frac{2}{p-1}$. It is then clear that as $\lambda \to \infty$:

(a) If $s > s_c$ (**sub-critical case**) the norm of the initial data can be made small while at the same time the time interval is made longer. This is the best possible scenario for local well–posedness. Notice that u^{λ} lives on $[0, \lambda^2 T]$.

(b) If $s = s_c$ (**critical case**) the norm of the initial data is invariant while the time interval gets longer. There is still hope in this case, but it turns out that to provide globally defined solutions one has to work very hard.

(c) If $s < s_c$ (**super-critical case**) the norms grow as the time interval is made longer. Scaling works against us in this case; we cannot expect even locally defined strong solutions, at least in deterministic sense.

(2) Then we have the **Galilean Invariance**: If u is a solution to (2.1) then

$$e^{ix\cdot v} e^{-it|v|^2} u(x - 2vt,\ t)$$

is a solution to the same equation with data $e^{ix\cdot v} u_0(x)$.

(3) Other symmetries:

(a) There is also **time reversal symmetry**. We can thus consider solutions in $[0, T]$ instead of $[-T, T]$.

(b) **Spatial rotation symmetry** which leads to the property that if we start with radial initial data then we obtain a radially symmetric solution.

(c) **Time translation invariance** that leads for smooth solutions to the conservation of energy

$$E(u)(t) = \frac{1}{2} \int |\nabla u(t)|^2 dx - \frac{\lambda}{p+1} \int |u(t)|^{p+1} dx = E(u_0). \tag{2.2}$$

(d) **Phase rotation symmetry** $e^{i\theta} u$ that leads to mass conservation

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{2.3}$$

(e) **Space translation invariance** that leads to the conservation of the momentum

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \bar{u} \nabla u dx = \vec{p}(0). \tag{2.4}$$

(4) In the case that $p = 1+\frac{4}{n}$, we also have the **pseudo-conformal symmetry** where if u is a solution to (2.1) then for $t \neq 0$

$$\frac{1}{|t|^{\frac{n}{2}}}\overline{u(\frac{x}{t},\frac{1}{t})}e^{\frac{i|x|^2}{4t}}$$

is also a solution. This leads to the pseudo-conformal conservation law

$$K(t) = \|(x+2it\nabla)u\|_{L^2}^2 - \frac{8t^2\lambda}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx = \|xu_0\|_{L^2}^2.$$

2.2. **Questions of interest and relevant notation.** We will study NLS and related equations via considering questions

- of local-in-time nature (local existence of solutions, uniqueness, regularity),
- of global-in-time nature (existence of solutions for large times, finite time blow-up, scattering).

The standard treatment of the subject is presented in the books of Cazenave [11] and Tao [71], among others. We will refer to these books, especially the first one, throughout the notes.

We start by listing some questions of interest:

1. Consider X a Banach space. Starting with initial data $u_0 \in H^s(\mathbb{R}^n)$, we say that the solution exists locally-in-time, if there exists $T > 0$ and a subset X of $C_t^0H_x^s([0,T]\times\mathbb{R}^n)$ such that there exists a unique solution to (2.1). Note that if $u(x,t)$ is a solution to (2.1) then $-u(-x,t)$ is also a solution. Thus we can extend any solution in $C_t^0H_x^s([0,T]\times\mathbb{R}^n)$ to a solution in $C_t^0H_x^s([-T,T]\times\mathbb{R}^n)$. We also demand that there is continuity with respect to the initial data in the appropriate topology.

2. If T can be taken to be arbitrarily large then we say that we have a global solution.

3. Assume $u_0 \in H^s(\mathbb{R}^n)$ and consider a local solution. If there is a $T^\star$ such that

$$\lim_{t\to T^\star}\|u(t)\|_{H^s} = \infty,$$

we say that the solution blows up in finite time. At this point, we can mention a statement of the so called "blow-up alternative" which is usually proved along with the local theory. More precisely, the blow-up alternative is a statement that characterizes the finite time of blow-up, which for example can be done along the following lines: if $(0,T^*)$ is the maximum interval of existence, then if $T^* < \infty$, we have $\lim_{t\to T^*}\|u(t)\|_{H^s} = \infty$. Analogous statements can be made for $(-T^*,0)$.

4. As a Corollary to the blow-up alternative one obtains globally defined solutions if there is an a priori bound of the H^s norms for all times. Such an a priori bound is of the form:

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{H^s} < \infty,$$

and it usually comes from the conservation laws of the equation. For (2.1) this is usually the case for $s = 0,\ 1$. An *important* comment is in order. Our notion of global solutions in the remark no. 2, described above, does not require that $\|u(t)\|_{H^s}$ remains uniformly bounded in time. As we said unless $s = 0,\ 1$, it is not a triviality to obtain such a uniform bound. In case that we have quantum scattering, these uniform bounds are byproducts of the control we obtain on our solutions at infinity.

5. If $u_0 \in H^s(\mathbb{R}^n)$ and we have a well defined local solution, then for each $(0, T)$ we have that $u(t) \in H^s_x(\mathbb{R}^n)$. Persistence of regularity refers to the fact that if we consider $u_0 \in H^{s_1}(\mathbb{R}^n)$ with $s_1 > s$, then $u \in X \subset C^0_t H^{s_1}_x([0, T_1] \times \mathbb{R}^n)$, with $T_1 = T$. Notice that any H^{s_1} solution is in particular an H^s solution and thus $(0, T_1) \subset (0, T)$. Persistence of regularity affirms that $T_1 = T$ and thus u cannot blow-up in H^{s_1} before it blows-up in H^s both backward and forward in time.

6. Scattering is usually the most difficult problem of the ones mentioned above. Assume that we have a globally defined solution (which is true for arbitrary large data in the defocusing case). The problem then is divided into an easier (existence of the wave operator) and a harder (asymptotic completeness) problem. We will see shortly that the L^p norms of linear solutions decay in time. This time decay is suggestive that for large values of p the nonlinearity can become negligible as $t \to \pm\infty$. Thus we expect that u can be approximated by the solution of the linear equation. We have to add here that this theory is highly nontrivial for large data. For small data we can have global solutions and scattering even in the focusing problem.

7. A solution that will satisfy (at least locally) most of these properties will be called a **strong** solution. We will give a more precise definition later in the notes. This is a distinction that is useful as one can usually derive through compactness arguments weak solutions that are not unique. The equipment of the derived (strong) solutions with the aforementioned properties is of importance. For example the fact that local H^1 solutions satisfy the energy conservation law is a byproduct not only of the local-in-time existence but also of the regularity and the continuity with respect to the initial data properties.

8. To make the exposition easier we mainly consider H^s solutions where s is an integer. From a mathematical point of view one can investigate solutions that evolve from rougher and rougher initial data (and thus belong to larger classes of spaces).

3. Local Well–Posedness

When trying to establish existence of local (in time) solutions, an important step consists of constructing the aforementioned Banach space X. This process is delicate (the exception being the construction of smooth solutions that is done classically) and is built upon certain estimates that the linear solution satisfies. First we recall those estimates.

3.1. **Fundamental solution, Dispersive and Strichartz estimates.**

Recall (from an undergraduate or graduate PDE course) that we can obtain the solution to the linear problem by utilizing the Fourier transform. Then for smooth initial data (say in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$) the solution of the linear homogeneous equation is given as the convolution of the data with the tempered distribution

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{i\frac{i|x|^2}{4t}}.$$

Thus we can write the linear solution as:

$$u(x,t) = U(t)u_0(x) = e^{it\Delta}u_0(x) = K_t \star u_0(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy. \quad (3.1)$$

Another fact from our undergraduate (or graduate) machinery is Duhamel's principle:
Let I be any time interval and suppose that $u \in C_t^1\mathcal{S}(I \times \mathbb{R}^n)$ and that $F \in C_t^0\mathcal{S}(I \times \mathbb{R}^n)$. Then u solves

$$\begin{cases} iu_t + \Delta u = F, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,t_0) = u(t_0) \in \mathcal{S}(\mathbb{R}^n) \end{cases} \quad (3.2)$$

if and only if

$$u(x,t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_0^t e^{i(t-s)\Delta}F(s)ds. \quad (3.3)$$

Definition 3.1. *Let I be a time interval which contains zero, $u_0 := u(x,0) \in H^s(\mathbb{R}^n)$ and*

$$F \in C(H^s(\mathbb{R}^n); H^{s-2}(\mathbb{R}^n)).$$

We say that

$$u \in C(I; H^s(\mathbb{R}^n)) \cap C^1(I; H^{s-2}(\mathbb{R}^n))$$

is a **strong solution** *of* (3.2) *on I, if it satisfies the equation for all $t \in I$ in the sense of H^{s-2} (thus as a distribution for low values of s) and $u(0) = u_0$.*

Remark 3.2. *By a little semigroup theory this definition of a strong solution is equivalent to saying that for all $t \in I$, u satisfies* (3.3).

Now we state the basic dispersive estimate for solutions to the homogeneous equation (3.2), with $F = 0$. From the formula (3.1) we see that:

$$\|u\|_{L_x^\infty} \leq \frac{1}{(4|t|\pi)^{\frac{n}{2}}}\|u_0\|_{L^1}.$$

In addition the solution satisfies that $\hat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\hat{u}_0(\xi)$, which together with Plancherel's theorem implies that

$$\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2}.$$

Riesz-Thorin interpolation Lemma then implies that for any $p \geq 2$ and $t \neq 0$ we have that

$$\|u(t)\|_{L_x^p} \leq \frac{1}{(4|t|\pi)^{n(\frac{1}{2}-\frac{1}{p})}}\|u_0\|_{L^{p'}}, \quad (3.4)$$

where p' is the dual exponent of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

Fortunately, the basic dispersive estimates (3.4) can be extended by duality (using a $TT^\star$ argument) to obtain very useful Strichartz estimates, [11, 31, 45, 58]. In order to state Strichartz estimates, first, we recall the definition of an admissible pair of exponents.

Definition 3.3. *Let $n \geq 1$. We call a pair (q, r) of exponents admissible if*

$$2 \leq q, r \leq \infty$$

are such that

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} \tag{3.5}$$

and $(q, r, n) \neq (2, \infty, 2)$.

Now we can state the Strichartz estimates:

Theorem 3.4. [31, 45] *Let $n \geq 1$. Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have the following estimates:*

- *The homogeneous estimate:*

$$\|e^{it\Delta}u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2}, \tag{3.6}$$

- *The dual estimate:*

$$\|\int_{\mathbb{R}} e^{-it\Delta}F(\cdot, t)\,dt\|_{L^2_x(\mathbb{R}^n)} \leq \|F\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x(\mathbb{R}\times\mathbb{R}^n)} \tag{3.7}$$

- *The non-homogeneous estimate:*

$$\|\int_0^t e^{i(t-s)\Delta}F(\cdot, s)\,ds\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.8}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

Remark 3.5. *Actually the proof of Strichartz estimates implies more. In particular, the operator $e^{it\Delta}u_0(x)$ belongs to $C(\mathbb{R}, L^2_x)$ and $\int_0^t U(t-s)F(s)ds$ belongs to $C(\bar{I}, L^2_x)$ where $t \in I$ is any interval of $\mathbb{R}$.*

3.2. **Notion of local well–posedness.** We are now ready to give a precise definition of what we mean by local well–posedness of the initial value problem (IVP) (2.1).

Definition 3.6. *We say that the IVP* (2.1) *is locally well–posed (lwp) and admits a strong solution in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$, there exists a finite time T and a Banach space $X \subset L^\infty_t H^s_x([0, T] \times \mathbb{R}^n)$ such that for any initial data $u_0 \in B$ there exists a unique solution $u \in X \subset C^0_t H^s_x([0, T] \times \mathbb{R}^n)$ to the integral equation*

$$u(x, t) = U(t)u_0 + i\lambda \int_0^t U(t-s)|u|^{p-1}u(s)ds.$$

Furthermore the map $u_0 \to u(t)$ is continuous as a map from $H^s(\mathbb{R}^n)$ into $C^0_t H^s_x([0, T] \times \mathbb{R}^n)$. If uniqueness holds in the whole space $C^0_t H^s_x([0, T] \times \mathbb{R}^n)$ then we say that the lwp is unconditional.

In what follows we assume that $p - 1 = 2k$. This implies that the nonlinearity is sufficiently smooth to perform all the calculations in a straightforward way.

3.3. **Well–posedness for smooth solutions.** We start with the H^s well-posedness theory, with an integer $s > \frac{n}{2}$. For more general statements see [44].

Theorem 3.7. *Let $s > \frac{n}{2}$ be an integer. For every $u_0 \in H^s(\mathbb{R}^n)$ there exists $T^* > 0$ and a unique maximal solution $u \in C((0, T^*); H^s(\mathbb{R}^n))$ that satisfies (2.1) and in addition satisfies the following properties:*
i) If $T^* < \infty$ then $\|u(t)\|_{H^s} \to \infty$ as $t \to \infty$. Moreover $\limsup_{t\to T^*} \|u(t)\|_{L^\infty} = \infty$.
ii) u depends continuously on the initial data in the following sense. If $u_{n,0} \to u_0$ in H^s and if u_n is the corresponding maximal solution with initial data $u_{n,0}$, then $u_n \to u$ in $L^\infty((0, T); H^s(\mathbb{R}^n))$ for every interval $[0, T] \subset [0, T^*)$.
iii) In addition, the solution u satisfies conservation of energy (4.2) and conservation of mass (4.3).

Remark 3.8. *A comment about uniqueness. Suppose that one proves existence and uniqueness in $C([-T, T]; X_M)$ where X_M, $M = M(\|u_0\|_X)$, $T = T(M)$, is a fixed ball in the space X. One can then easily extend the uniqueness to the whole space X by shrinking time by a fixed amount. Indeed, shrinking time to T' we get existence and uniqueness in a larger ball $X_{M'}$. Now assume that there are two different solutions one staying in the ball X_M and one separating after hitting the boundary at some time $|t| < T'$. This is already a contradiction by the uniqueness in $X_{M'}$.*

3.3.1. *Preliminaries.* To prove Theorem 3.7 we need the following two lemmata:

Lemma 3.9. *Gronwall's inequality: Let $T > 0$, $k \in L^1(0, T)$ with $k \geq 0$ a.e. and two constants $C_1, C_2 \geq 0$. If $\psi \geq 0$, a.e in $L^1(0, T)$, such that $k\psi \in L^1(0, T)$ satisfies*

$$\psi(t) \leq C_1 + C_2 \int_0^t k(s)\psi(s)ds$$

for a.e. $t \in (0, T)$ then,

$$\psi(t) \leq C_1 \exp\big(C_2 \int_0^t k(s)ds\big).$$

Proof. For a proof, see e.g. Evans [28]. □

Lemma 3.10. *Let $g(u) = \pm|u|^{2k}u$ and consider and $s, l \geq 0$, integers with $l \leq s$ and $s > \frac{n}{2}$. Then*

$$\|g(u)\|_{H^s} \lesssim \|u\|_{H^s}^{2k+1}, \tag{3.9}$$

$$\|g(u) - g(v)\|_{L^2} \lesssim \big(\|u\|_{H^s}^{2k} + \|v\|_{H^s}^{2k}\big)\|u - v\|_{L^2}, \tag{3.10}$$

$$\|g^{(l)}(u) - g^{(l)}(v)\|_{L^\infty} \lesssim \big(\|u\|_{H^s}^{2k-l} + \|v\|_{H^s}^{2k-l}\big)\|u - v\|_{H^s}, \tag{3.11}$$

$$\|g(u) - g(v)\|_{H^s} \lesssim \big(\|u\|_{H^s}^{2k} + \|v\|_{H^s}^{2k}\big)\|u - v\|_{H^s}. \tag{3.12}$$

Proof. To prove (3.9) we use the algebra property of H^s for $s > \frac{n}{2}$ and the fact that $\|u\|_{H^s} = \|\bar{u}\|_{H^s}$.

To prove (3.10) and (3.11) note that since g is smooth we have that

$$|g(u) - g(v)| \lesssim \big(|u|^{2k} + |v|^{2k}\big)|u - v|,$$

$$|g^{(l)}(u) - g^{(l)}(v)| \lesssim \big(|u|^{2k-l} + |v|^{2k-l}\big)|u - v|.$$

Then

$$\|g(u) - g(v)\|_{L^2} \lesssim \big(\|u\|_{L^\infty}^{2k} + \|v\|_{L^\infty}^{2k}\big)\|u - v\|_{L^2} \lesssim \big(\|u\|_{H^s}^{2k} + \|v\|_{H^s}^{2k}\big)\|u - v\|_{L^2},$$

$$\|g^{(l)}(u)-g^{(l)}(v)\|_{L^\infty} \lesssim \big(\|u\|_{L^\infty}^{2k-l}+\|v\|_{L^\infty}^{2k-l}\big)\|u-v\|_{L^\infty} \lesssim \big(\|u\|_{H^s}^{2k-l}+\|v\|_{H^s}^{2k-l}\big)\|u-v\|_{L^2},$$

where we used the fact that H^s embeds in L^∞.

To prove (3.12) notice that the L^2 part of the left hand side follows from (3.10). For the derivative part consider a multi-index α with $|\alpha| = s$. Then $D^\alpha u$ is the sum (over $k \in \{1, 2, ..., s\}$) of terms of the form $g^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u$ where $|\beta_j| \geq 1$ and $|\alpha| = |\beta_1| + ... + |\beta_k|$. Now let $p_j = \frac{2s}{|\beta_j|}$ such that $\sum_{j=1}^k \frac{1}{p_j} = \frac{1}{2}$. We have by Hölder's inequality

$$\|g^{(k)}(u) \prod_{j=1}^{k} D^{\beta_j} u\|_{L^2} \lesssim \|g^{(k)}(u)\|_{L^\infty} \prod_{j=1}^{k} \|D^{\beta_j} u\|_{L^{p_j}}.$$

By complex interpolation (or Gagliardo-Nirenberg inequality) we obtain

$$\|D^{\beta_j} u\|_{L^{p_j}} \lesssim \|u\|_{H^s}^{\frac{|\beta_j|}{s}} \|u\|_{L^\infty}^{1-\frac{|\beta_j|}{s}}$$

and thus

$$\|g^{(k)}(u) \prod_{j=1}^{k} D^{\beta_j} u\|_{L^2} \lesssim \|g^{(k)}(u)\|_{L^\infty} \|u\|_{H^s} \|u\|_{L^\infty}^{k-1} \lesssim \|u\|_{H^s}^{2k+1}$$

where in the last inequality we used (3.11). Thus we obtain

$$\|D^\alpha u\|_{L^2} \lesssim \|u\|_{H^s}^{2k+1}. \tag{3.13}$$

Again notice that the term $D^\alpha(g(u) - g(v))$ is the sum of terms of the form

$$g^{(k)}(u) \prod_{j=1}^{k} D^{\beta_j} u - g^{(k)}(v) \prod_{j=1}^{k} D^{\beta_j} v = \big[g^{(k)}(u) - g^{(k)}(v)\big] \prod_{j=1}^{k} D^{\beta_j} u + g^{(k)}(v) \prod_{j=1}^{k} D^{\beta_j} w_j$$

where w_j's are equal to u or v except one that is equal to $u - v$. The second of the left hand side is estimated as in the proof of (3.13). For the first the same trick applies but now to estimate $\|g^{(k)}(u) - g^{(k)}(v)\|_{L^\infty}$ we use (3.12). □

3.3.2. *A proof of Theorem 3.7.* Now we present a proof of Theorem 3.7.

Existence and Uniqueness. We construct solutions by a fixed point argument.

Given $M, T > 0$ to be chosen later, we set $I = (0, T)$ and consider the space

$$E = \{u \in L^\infty(I; H^s(\mathbb{R}^n)) : \|u\|_{L^\infty(I;H^s)} \leq M\},$$

equipped with the distance

$$d(u, v) = \|u - v\|_{L^\infty(I;L^2)}.$$

We note that (E, d) is a complete metric space.

Now based on the equation (2.1), with $\lambda = -1$, in the integral form, we introduce the mapping Φ as follows:

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}|u|^{2k}u(\tau)\,d\tau =: e^{it\Delta}u_0 + H(u)(t).$$

By Lemma 3.10, Minkowski's inequality and the fact that $e^{it\Delta}$ is an isometry in H^s we have that

$$\|\Phi(u)(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + T\|g(u)\|_{L^\infty(I;H^s)} \leq \|u_0\|_{H^s} + TC(M)M,$$

where we used the notation $g(u) = \pm|u|^{2k}u$ as in Lemma 3.10. Furthermore using Lemma 3.10 again we have

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L^2} \lesssim TC(M)\|u - v\|_{L^\infty(I;L^2)}. \tag{3.14}$$

Therefore we see that if $M = 2\|u_0\|_{H^s}$ and $TC(M) < \frac{1}{2}$, then Φ is a contraction of (E, d) and thus has a unique fixed point. Uniqueness in the full space follows by the remark above or alternatively by the remark and Gronwall's Lemma.

Blow-up alternative. Let $u_0 \in H^s$ and define

$$T^* = \sup\{T > 0 : \textit{there exists a solution on } [0,T]\}. \tag{3.15}$$

Now let $T^* < \infty$ and assume that there exists a sequence $t_j \to T^*$ such that $\|u(t_j)\|_{H^s} \leq M$. In particular for k such that t_k is close to T^* we have that $\|u(t_k)\|_{H^s} \leq M$. Now we solve our problem with initial data $u(t_k)$ and we extend our solution to the interval $[t_k,\ t_k + T(M)]$. But if we pick k such that

$$t_k + T(M) > T^*$$

we then contradict the definition of T^*. Thus $\lim_{t\to T^*}\|u(t)\|_{H^s} = \infty$ if $T^* < \infty$.

We now show that if $T^* < \infty$ then $\limsup_{t\to T^*}\|u(t)\|_{L^\infty} = \infty$. Indeed suppose that $\limsup_{t\to T^*}\|u(t)\|_{L^\infty} < \infty$. Since $u \in C([0,T^*);H^s)$ we have that

$$M = \sup_{0\leq t<T^*}\|u(t)\|_{L^\infty} < \infty$$

where we used the fact that H^s embeds in L^∞. By Duhamel's formula and Lemma 3.10 we have that

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C(M)\int_0^t \|u(\tau)\|_{H^s}\,d\tau.$$

By Gronwall's lemma we have that $\|u(t)\|_{H^s} \leq \|u_0\|_{H^s}e^{T^*C(M)}$ for all $0 \leq t < T^*$. But this contradicts the blow-up of $\|u(t)\|_{H^s}$ at T^*.

Continuous dependence. Let $u_0 \in H^s$ and consider $u_{0,n} \subset H^s$ such that $u_{n,0} \to u_0$ in H^s as $n \to \infty$. Since for n sufficiently large we have that $\|u_{0,n}\|_{H^s} \leq 2\|u_0\|_{H^s}$ by the local theory there exists $T = T(\|u_0\|_{H^s})$ such that u and u_n are defined on $[0,T]$ for $n \geq N$ and

$$\|u\|_{L^\infty((0,T);H^s)} + \sup_{n\geq N}\|u_n\|_{L^\infty((0,T);H^s)} \leq 6\|u_0\|_{H^s}.$$

Now note that $u_n(t)-u(t)=e^{it\Delta}(u_{n,0}-u_0)+H(u_n)(t)-H(u)(t)$. If we use Lemma 3.10 we see that for all $t\in(0,T)$ and n sufficiently large, there exists C such that

$$\|u_n(t)-u(t)\|_{H^s}\le\|u_{n,0}-u_0\|_{H^s}+C\int_0^t\|u_n(\tau)-u(\tau)\|_{H^s}\,d\tau.$$

By Gronwall's lemma we see that $u_n\to u$ in H^s as $n\to\infty$. Iterating this property to cover any compact subset of $(0,T^*)$ we finish the proof.

As a final note we remark that if we solve the equation, starting from u_0 and $u(t_1)$ over the intervals $[0,t_1]$ and $[t_1,t_2]$ respectively, by continuous dependence, to prove that $C\big([0,T];H^s(\mathbb{R}^n)\big)$, it is enough to consider the difference $u(t_1)-u_0$ in the H^s norm. Since

$$u(t_1)-u_0=(e^{it_1\Delta}-1)u_0-i\int_0^{t_1}e^{i(t_1-\tau)}g(u)(\tau)\,d\tau,$$

using again Lemma 3.10 and the fact that $e^{it\Delta}u_0(x)\in C(\mathbb{R};H^s)$ we have

$$\|u(t_1)-u_0\|_{H^s}\lesssim\|(e^{it_1\Delta}-1)u_0\|_{H^s}+|t_1|\|u\|^{2k+1}_{L^\infty((0,t_1);H^s)}$$

which finishes the proof.

<u>*Conservation laws:*</u> Since we develop the H^1 theory below we implicitly have $s\ge 2$. We have at hand a solution that satisfies the equation in the classical sense for high enough s (in general in the H^{s-2} sense with $s\ge2$ and thus in particular u satisfies the equation at least in the L^2 sense. All integrations below then can be justified in the Hilbert space L^2). To obtain the conservation of mass we can multiply the equation by $i\bar{u}$, integrate and then take the real part. To obtain the conservation of energy we multiply the equation by $\bar{u}_t$, take the real part and then integrate.

3.4. **Local well–posedness in the H^1 sub–critical case.** For more details we refer to [11, 43, 44].

Theorem 3.11. *Let $1<p<1+\frac{4}{n-2}$, if $n\ge3$ and $1<p<\infty$, if $n=1,2$. For every $u_0\in H^1(\mathbb{R}^n)$ there exists a unique strong H^1 solution of* (2.1) *defined on the maximal interval $(0,T^\star)$. Moreover*

$$u\in L^\gamma_{loc}((0,T^\star);\ W^{1,\rho}_x(\mathbb{R}^n))$$

for every admissible pair (γ,ρ). In addition

$$\lim_{t\to T^\star}\|u(t)\|_{H^1}=\infty$$

if $T^\star<\infty$, and u depends continuously on u_0 in the following sense: There exists $T>0$ depending on $\|u_0\|_{H^1}$ such that if $u_{0,n}\to u_0$ in H^1 and $u_n(t)$ is the corresponding solution of (2.1)*, then $u_n(t)$ is defined on $[0,T]$ for n sufficiently large and*

$$u_n(t)\to u(t)\quad in\quad C([0,T];\ H^1)\tag{3.16}$$

for every compact interval $[0,T]$ of $(0,T^\star)$. Finally we have that

$$E(u)(t)=\frac12\int|\nabla u(t)|^2dx-\frac{\lambda}{p+1}\int|u(t)|^{p+1}dx=E(u_0)$$

and

$$M(u)(t)=\|u(t)\|_{L^2}=\|u_0\|_{L^2}=M(u_0).$$

We note that $W^{1,\rho}$ is the Sobolev space of L^ρ functions with weak derivatives in L^ρ of order one.

Proof. First we establish:
Existence and Uniqueness. In order to define the space on which we shall apply the fixed point argument, we pick r to be $r := p+1$. Fix $M, T > 0$ to be chosen later and let q be such that the pair (q, r) is admissible.[3] Consider the set

$$E = \{u \in L_t^\infty H_x^1([0,T] \times \mathbb{R}^n) \cap L^q((0,T); W^{1,r}(\mathbb{R}^n)) : \tag{3.17}$$

$$\|u\|_{L_t^\infty((0,T);H^1)} \leq M \text{ and } \|u\|_{L_t^q W_x^{1,r}} \leq M\}. \tag{3.18}$$

equipped with the distance

$$d(u,v) = \|u-v\|_{L^q((0,T);L^r(\mathbb{R}^n))} + \|u-v\|_{L^\infty((0,T);L^2(\mathbb{R}^n))}.$$

It can be shown that (E, d) is a complete metric space.

We write the solution map via Duhamel's formula as follows:

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}|u|^{p-1}u(\tau)\,d\tau. \tag{3.19}$$

Now we provide a few estimates that we shall use in order to justify that the mapping Φ is a contraction on (E, d). Notice that for $r = p+1$ we have

$$\||u|^{p-1}u\|_{L_x^{r'}} \lesssim \|u\|_{L_x^r}^p$$

and thus by Hölder

$$\||u|^{p-1}u\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty L_x^r}^{p-1}\|u\|_{L_t^q L_x^r}. \tag{3.20}$$

However by Sobolev embedding we have that

$$\|u\|_{L_x^r} \lesssim \|u\|_{H^1},$$

which together with (3.20) implies that:

$$\||u|^{p-1}u\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1}\|u\|_{L_t^q L_x^r}. \tag{3.21}$$

Similarly, since the nonlinearity is smooth,

$$\|\nabla(|u|^{p-1}u)\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1}\|\nabla u\|_{L_t^q L_x^r}. \tag{3.22}$$

Now we combine (3.21) and (3.22) to obtain for $u \in E$:

$$\||u|^{p-1}u\|_{L_t^q W_x^{1,r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1}\|u\|_{L_t^q W_x^{1,r}} \tag{3.23}$$

Furthermore, applying Hölder's inequality in time, followed by an application of (3.23) gives:

$$\begin{aligned}\||u|^{p-1}u\|_{L_t^{q'} W_x^{1,r'}} &\lesssim T^{\frac{q-q'}{q'q}}\||u|^{p-1}u\|_{L_t^q W_x^{1,r'}} \\ &\lesssim T^{\frac{q-q'}{q'q}}\|u\|_{L_t^\infty H_x^1}^{p-1}\|u\|_{L_t^q W_x^{1,r}}. \end{aligned} \tag{3.24}$$

[3]Since the admissibility condition reads $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, and $r = p+1$, we have that $q = \frac{4(p+1)}{n(p-1)}$.

Now we are ready to show that Φ is a contraction on (E,d). Using Duhamel's formula (3.19) and Strichartz estimates we obtain:

$$\|\Phi(u)(t)\|_{L^q_tW^{1,r}_x} \lesssim \|e^{it\Delta}u_0\|_{L^q_tW^{1,r}_x} + \||u|^{p-1}u\|_{L^{q'}_tW^{1,r'}_x}$$
$$\lesssim \|u_0\|_{H^1} + T^{\frac{q-q'}{q'q}}\|u\|^{p-1}_{L^\infty_tH^1_x}\|u\|_{L^q_tW^{1,r}_x}, \tag{3.25}$$

where to obtain (3.25) we used (3.24). Also by Duhamel's formula (3.19), Strichartz estimates and (3.24) we have:

$$\|\Phi(u)(t)\|_{L^\infty_tH^1_x} \lesssim \|u_0\|_{H^1} + T^{\frac{q-q'}{q'q}}\|u\|^{p-1}_{L^\infty_tH^1_x}\|u\|_{L^q_tW^{1,r}_x}. \tag{3.26}$$

Hence (3.25) and (3.26) imply:

$$\|\Phi(u)(t)\|_{L^q_tW^{1,r}_x} + \|\Phi(u)(t)\|_{L^\infty_tH^1_x} \leq C\|u_0\|_{H^1} + CT^{\frac{q-q'}{q'q}}M^{p-1}\|u\|_{L^q_tW^{1,r}_x}. \tag{3.27}$$

Now we set $M = 2C\|u_0\|_{H^1}$ and then choose T small enough such that

$$CT^{\frac{q-q'}{q'q}}M^{p-1} \leq \frac{1}{2}.$$

We note that such choice of T is indeed possible thanks to the fact that for $p < 1+\frac{4}{n-2}$ we have that $q > 2$ and thus $q > q'$. For such $T \sim T(\|u_0\|_{H^1})$ we have that $\|\Phi(u)(t)\|_E \leq M$ whenever $u \in E$ and thus $\Phi : E \to E$. In a similar way, one can obtain the following estimate on the difference:

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L^q_tW^{1,r}_x} + \|\Phi(u)(t) - \Phi(v)(t)\|_{L^\infty_tL^2_x}$$

provides a unique solution $u \in E$. Notice that by the above estimates and the Strichartz estimates we have that $u \in C^0_t((0,T);H^1(\mathbb{R}^n))$.

To extend uniqueness in the full space we assume that we have another solution v and consider an interval $[0,\delta]$ with $\delta < T$. Then as before

$$\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x} \leq C\delta^\alpha(\|u\|^{p-1}_{L^\infty_TH^1_x} + \|v\|^{p-1}_{L^\infty_TH^1_x})\|u-v\|_{L^q_\delta W^{1,r}_x}$$

But if we set

$$K = \max(\|u\|_{L^\infty_TH^1_x} + \|v\|_{L^\infty_TH^1_x}) < \infty$$

then for δ small enough we obtain

$$\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x} \leq \frac{1}{2}(\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x})$$

which forces $u = v$ on $[0,\delta]$. To cover the whole $[0,T]$ we iterate the previous argument $\frac{T}{\delta}$ times.

Membership in the Strichartz space. The fact that

$$u \in L^\gamma_{loc}((0,T^*);\ W^{1,\rho}_x(\mathbb{R}^n))$$

for every admissible pair (γ,ρ), follows from the Strichartz estimates on any compact interval inside $(0,T^*)$.

Blow-up alternative. The proof is the same as in the smooth case.

Continuous dependence can be obtained via establishing estimates on

$$\|u_n(t) - u(t)\|_{L^q_tW^{1,r}_x} + \|u_n(t) - u(t)\|_{L^\infty_tH^1_x}.$$

We skip details and refer the interested reader to [11].

Conservation laws. The proof of the conservation of mass is similar to the smooth case but now we use the pairing $(u_t, u)_{H^1-H^{-1}}$. A proof of conservation of energy is more involved since we need more derivatives to make sense of the energy functional. Details can be found in e.g. [11]. □

Remark 3.12. *We pause to give a couple of comments:*

(1) *Notice that when $\lambda = -1$ (defocusing case), the mass and energy conservation provide a global a priori bound*
$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \le C_{M(u_0),E(u_0)}.$$
By the blow-up alternative we then have that $T^* = \infty$ and the problem is globally well-posed (gwp).

(2) *Let $I = [0, T]$. An inspection of the proof reveals that we can run the lwp argument in the space $\mathcal{S}^1(I \times \mathbb{R}^n)$ with the norm*
$$\|u\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$
where
$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible} \|u\|_{L^q_{t\in I}L^r_x}.$$

3.5. **Well–posedness for the L^2 sub–critical problem.** We now state the lwp and gwp theory for the L^2 sub–critical problem. The reader can consult e.g. [74] for details.

Theorem 3.13. *Consider $1 < p < 1 + \frac{4}{n}$, $n \ge 1$ and an admissible pair (q, r) with $p + 1 < q$. Then for every $u_0 \in L^2(\mathbb{R}^n)$ there exists a unique strong solution of*
$$\begin{cases} iu_t + \Delta u + \lambda|u|^{p-1}u = 0, \\ u(x, 0) = u_0(x) \end{cases} \tag{3.28}$$
defined on the maximal interval $(0, T^\star)$ such that
$$u \in C^0_t((0, T^\star); L^2(\mathbb{R}^n)) \cap L^q_{loc}((0, T^\star); L^r(\mathbb{R}^n)).$$
Moreover
$$u \in L^\gamma_{loc}((0, T^\star); L^\rho(\mathbb{R}^n))$$
for every admissible pair (γ, ρ). In addition
$$\lim_{t\to T^\star} \|u(t)\|_{L^2} = \infty$$
if $T^\star < \infty$ and u depends continuously on u_0 in the following sense: There exists $T > 0$ depending on $\|u_0\|_{L^2}$ such that if $u_{0,n} \to u_0$ in L^2 and $u_n(t)$ is the corresponding solution of (3.28), then $u_n(t)$ is defined on $[0, T]$ for n sufficiently large and
$$u_n(t) \to u(t) \ \ in \ \ L^\gamma_{loc}([0, T];\ L^\rho(\mathbb{R}^n)) \tag{3.29}$$
for every admissible pair (γ, ρ) and every compact interval $[0, T]$ of $(0, T^\star)$. Finally we have that
$$M(u)(t) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0) \ \ \text{and thus } T^\star = \infty. \tag{3.30}$$

Remark 3.14. *We give a couple of comments:*

(1) *Notice that global well-posedness follows immediately.*
(2) *The equation makes sense in H^{-2}.*

Finally we state the L^2–critical lwp theory when $p = 1 + \frac{4}{n}$. We should mention that a similar theory holds for the H^1 critical problem ($p = 1 + \frac{4}{n-2}$), [12]. For dimensions $n = 1, 2$ the problem is always energy sub-critical.

Theorem 3.15. *Consider $p = 1 + \frac{4}{n}$, $n \geq 1$. Then for every $u_0 \in L^2(\mathbb{R}^n)$ there exists a unique strong solution of*

$$\begin{cases} iu_t + \Delta u + \lambda |u|^{\frac{4}{n}} u = 0, \\ \qquad u(x, 0) = u_0(x) \end{cases} \tag{3.31}$$

defined on the maximal interval $(0, T^\star)$ such that

$$u \in C_t^0((0, T^\star); L^2(\mathbb{R}^n)) \cap L_{loc}^{p+1}((0, T^\star); L^{p+1}(\mathbb{R}^n)).$$

Moreover

$$u \in L_{loc}^{\gamma}((0, T^\star); L^{\rho}(\mathbb{R}^n))$$

for every admissible pair (γ, ρ). In addition if $T^\star < \infty$

$$\lim_{t \to T^\star} \|u(t)\|_{L^q_{loc}((0,T^\star); L^r(\mathbb{R}^n))} = \infty$$

for every admissible pair (q, r) with $r \geq p + 1$. u also depends continuously on u_0 in the following sense: If $u_{0,n} \to u_0$ in L^2 and $u_n(t)$ is the corresponding solution of (3.31)*, then $u_n(t)$ is defined on $[0, T]$ for n sufficiently large and*

$$u_n(t) \to u(t) \quad in \quad L^q([0, T]);\ L^r(\mathbb{R}^n)) \tag{3.32}$$

for every admissible pair (q, r) and every compact interval $[0, T]$ of $(0, T^\star)$. Finally we have that

$$M(u)(t) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0) \quad for\ all\ t \in (0, T^\star). \tag{3.33}$$

Remark 3.16. *Again, we give a few comments:*

(1) *Notice that the blow-up alternative in this case is not in terms of the L^2 norm, which is the conserved quantity of the problem. This is because the problem is critical and the time of local well-posedness depends not only on the norm but also on the profile of the initial data. On the other hand if we have a global Strichartz bound on the solution global well-posedness is guaranteed by the Theorem. We will see later that this global Strichartz bound is sufficient for proving scattering also.*
(2) *It is easy to see that if $\|u_0\|_{L^2} < \mu$, for μ small enough, then by the Strichartz estimates*

$$\|e^{it\Delta} u_0\|_{L_t^{p+1} L_x^{p+1}(\mathbb{R} \times \mathbb{R}^n)} < C\mu < \eta.$$

Thus for sufficiently small initial data $T^\star = \infty$ and after only one iteration we have global well-posedness for the focusing or defocusing problem. In addition we have that $u \in L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^n))$ for every admissible pair (q, r) and thus we also have scattering for small data. But this is not true for large data as the following example shows.

Consider $\lambda > 0$. *We know that there exists nontrivial solutions of the form*
$$u(x,t) = e^{i\omega t}\phi(x)$$
where ϕ *is a smooth nonzero solution of*
$$-\Delta\phi + \omega\phi = |\phi|^{p-1}\phi$$
with $\omega > 0$. *But*
$$\|\phi\|_{L^r_x(\mathbb{R}^n)} \leq M$$
for every $r \geq 2$ *and thus* $u \notin L^q_t(\mathbb{R}; L^r_x(\mathbb{R}^n))$ *for any* $q < \infty$.

Although some recent results have appeared for super–critical equations, the theory has been completed only for the mass and energy critical problem and those developments are recent. More precisely, global energy solutions for the 3d defocusing energy-critical problem with radially symmetric initial data was obtained in [10]. The radially symmetric assumption was removed in [18]. For $n \geq 4$ the problem was solved in [65, 75]. The defocusing mass-critical problem is now solved in all dimensions in a series of papers, [20, 21, 22].

To obtain global-in-time solutions for the focusing problems, as we have seen, one needs to assume a bound on the norm of the data. For the energy-critical focusing problem one can consult the work [46], where a powerful program that helped settle many critical problems, has been introduced; for higher dimensions see e.g. [52]. Results concerning the mass-critical focusing problem are obtained in [23] in all dimensions.

4. Morawetz type inequalities

Consider the semi–linear Schrödinger equation (NLS) in arbitrary dimensions
$$\begin{cases} iu_t + \Delta u + \lambda|u|^{p-1}u = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad \lambda \pm 1, \\ \qquad u(x,0) = u_0(x) \end{cases} \tag{4.1}$$
for any $1 < p < \infty$.

Smooth solutions of the NLS equation satisfy energy
$$E(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |u(t)|^{p+1} dx = E(u_0) \tag{4.2}$$
and mass
$$M(u)(t) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0) \tag{4.3}$$
conservation.

We have seen the basic local well–posedness theory in the first part of this course. To study in more details the global solutions of the above problems we have to revisit the symmetries of the equation. We first write down the local conservation laws or the conservation laws in differentiable form. The differential form of the conservation law is more flexible and powerful as it can be localized to any given region of space–time by integrating against a suitable cut–off function or contracting against a suitable vector fields. One then does not obtain a conserved quantity but rather a monotone quantity. Thus from a single conservation law one can generate

a variety of useful estimates. We can also use these formulas to study the blow–up and concentration problems for the focusing NLS and the scattering problem for the defocusing NLS.
The question of scattering or in general the question of dispersion of the nonlinear solution is tied to weather there is some sort of decay in a certain norm, such as the L^p norm for $p > 2$. In particular knowing the exact rate of decay of various L^p norms for the linear solutions, it would be ideal to obtain estimates that establish similar rates of decay for the nonlinear problem. The decay of the linear solutions can immediately establish weak quantum scattering in the energy space but to estimate the linear and the nonlinear dynamics in the energy norm we usually looking for the L^p norm of the nonlinear solution to go to zero as $t \to \infty$.
Strichartz type estimates assure us that certain L^p norms going to zero but only for the linear part of the solution. For the nonlinear part we need to obtain general decay estimates on solutions of defocusing equations. The mass and energy conservation laws establish the boundedness of the L^2 and the H^1 norms but are insufficient to provide a decay for higher powers of Lebesgue norms. In these notes we provide a summary of recent results that demonstrate a straightforward method to obtain such estimates by taking advantage of the momentum conservation law

$$\Im \int_{\mathbb{R}^n} \overline{u} \nabla u dx = \Im \int_{\mathbb{R}^n} \overline{u_0} \nabla u_0 dx. \tag{4.4}$$

Thus we want to establish a priori estimates for the solutions to the power type nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1} u, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n) \end{cases} \tag{4.5}$$

for any $p > 1$ and $\lambda \in \mathbb{R}$. Equation (4.5) is the Euler-Lagrange equation for the Lagrangian density

$$L(u) = -\frac{1}{2}\Delta(|u|^2) + \lambda \frac{p-1}{p+1}|u|^{p+1}.$$

Space translation invariance leads to momentum conservation

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \bar{u} \nabla u dx, \tag{4.6}$$

a quantity that has no definite sign. It turns out that one can use this conservation law in the defocusing case and prove monotonicity formulas that are very useful in studying the global-in-time properties of the solutions at $t = \infty$. For most of these classical results the reader can consult [11], [71].

The study of the problem at infinity is an attempt to describe and classify the asymptotic behavior-in-time for the global solutions. To handle this issue, one tries to compare the given nonlinear dynamics with suitably chosen simpler asymptotic dynamics. For the semilinear problem (4.5), the first obvious candidate for the simplified asymptotic behavior is the free dynamics generated by the group $S(t) = e^{-it\Delta}$. The comparison between the two dynamics gives rise to the questions of the existence of wave operators and of the asymptotic completeness of the solutions. More precisely, we have:
i) Let $v_+(t) = S(t)u_+$ be the solution of the free equation. Does there exist a solution u of equation (4.5) which behaves asymptotically as v_+ as $t \to \infty$, typically in the sense that $\|u(t) - v_+\|_{H^1} \to 0$, as $t \to \infty$. If this is true, then one can define

the map $\Omega_+ : u_+ \to u(0)$. The map is called the wave operator and the problem of existence of u for given u_+ is referred to as the problem of the *existence of the wave operator*. The analogous problem arises as $t \to -\infty$.
ii) Conversely, given a solution u of (4.5), does there exist an asymptotic state u_+ such that $v_+(t) = S(t)u_+$ behaves asymptotically as $u(t)$, in the above sense. If that is the case for any u with initial data in X for some $u_+ \in X$, one says that *asymptotic completeness* holds in X.

In effect the existence of wave operators asks for the construction of global solutions that behave asymptotically as the solution of the free Schrödinger equation while the asymptotic completeness requires all solutions to behave asymptotically in this manner. It is thus not accidental that asymptotic completeness is a much harder problem than the existence of the wave operators (except in the case of small data theory which follows from the iterative methods of the local well-posedness theory).

Asymptotic completeness for large data not only require a repulsive nonlinearity but also some decay for the nonlinear solutions. A key example of these ideas is contained in the following generalized virial inequality, [54]:

$$\int_{\mathbb{R}^n \times \mathbb{R}} (-\Delta\Delta a(x))|u(x,t)|^2 dxdt + \lambda \int_{\mathbb{R}^n \times \mathbb{R}} (\Delta a(x))|u(x,t)|^{p+1} dxdt \leq C \qquad (4.7)$$

where $a(x)$ is a convex function, u is a solution to (4.5), and C a constant that depends only on the energy and mass bounds.

An inequality of this form, which we will call a one–particle inequality, was first derived in the context of the Klein-Gordon equation by Morawetz in [57], and then extended to the NLS equation in [54]. Most of these estimates are referred in the literature as Morawetz type estimates. The inequality was applied to prove asymptotic completeness first for the nonlinear Klein-Gordon and then for the NLS equation in [58], and then in [54] for slightly more regular solutions in space dimension $n \geq 3$. The case of general finite energy solutions for $n \geq 3$ was treated in [33] for the NLS and in [32] for the Hartree equation. The treatment was then improved to the more difficult case of low dimensions in [59, 60].

The bilinear a priori estimates that we outline here give stronger bounds on the solutions and in addition simplify the proofs of the results in the papers cited above. For a detailed summary of the method see [34]. In the original paper by Morawetz, the weight function that was used was $a(x) = |x|$. This choice has the advantage that the distribution $-\Delta\Delta(\frac{1}{|x|})$ is positive for $n \geq 3$. More precisely it is easy to compute that $\Delta a(x) = \frac{n-1}{|x|}$ and that

$$-\Delta\Delta a(x) = \begin{cases} 8\pi\delta(x), & \text{if} \quad n = 3 \\ \frac{(n-1)(n-3)}{|x|^2}, & \text{if} \quad n \geq 4. \end{cases}$$

In particular, the computation in (4.7) gives the following estimate for $n = 3$ and λ positive

$$\int_{\mathbb{R}} |u(t,0)|^2 dt + \int_{\mathbb{R}^3\times\mathbb{R}} \frac{|u(x,t)|^{p+1}}{|x|} dxdt \lesssim 1. \tag{4.8}$$

Similar estimates are true in higher dimensions. The second, nonlinear term, or certain local versions of it, have played central role in the scattering theory for the nonlinear Schrödinger equation, [10], [33], [37], [54]. The fact that in 3d, the bi–harmonic operator acting on the weight $a(x)$ produces the $\delta-$measure can be exploited further. In [17], a quadratic Morawetz inequality was proved by correlating two nonlinear densities $\rho_1(x) = |u(x)|^2$ and $\rho_2(y) = |u(y)|^2$ and define as $a(x, y)$ the distance between x and y in 3d. The authors obtained an a priori estimate of the form $\int_{\mathbb{R}^3\times\mathbb{R}} |u(x,t)|^4 dx \leq C$ for solutions that stay in the energy space. A frequency localized version of this estimate has been successfully implemented to remove the radial assumption of Bourgain, [10], and prove global well-posedness and scattering for the energy-critical (quintic) equation in 3d, [18]. For $n \geq 4$ new quadratic Morawetz estimates were given in [72]. Finally in [14] and in [64] these estimates were extended to all dimensions.

We should mention that taking as the weight function the distance between two points in $\mathbb{R}^n$ is not the only approach, see [15] for a recent example. Nowadays it is well understood that the bilinear Morawetz inequalities provide a unified approach for proving energy scattering for energy sub-critical solutions of the NLS when $p > 1 + \frac{4}{n}$ (L^2 super-critical nonlinearities). This last statement has been rigorously formalized only recently due to the work of the aforementioned authors, and a general exposition has been published in [34]. Infinite–energy solution scattering in the same range of powers has been initiated in [17]. For the L^2–critical problem, scattering is a very hard problem, but the problem has now been resolved in a series of new papers by B. Dodson, [20, 21, 22]. For mass sub–critical solutions, scattering even in the energy space is a very hard problem, and is probably false. Nevertheless, two particle Morawetz estimates have been used for the problem of the existence (but not uniqueness) of the wave operator for mass subcritical problems, [42]. We have already mentioned their implementation to the hard problem of energy critical solutions in [10], [37], and [18]. Recent results have used these inequalities for the mass critical problem, [20], and the energy super-critical problem, [53]. For a frequency localized one particle Morawetz inequality and its application to the scattering problem for the mass–critical equation with radial data see [73].

We are now ready to derive the estimates. As we mentioned we start with the equation

$$iu_t + \Delta u = \lambda |u|^{p-1} u \tag{4.9}$$

with $p \geq 1$ and $\lambda \in \mathbb{R}$. We use Einstein's summation convention throughout. According to this convention, when an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values. We will also write $\nabla_j u$ for $\frac{\partial u}{\partial x_j}$. For a function $a(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ we define $\nabla_{x,j}\ a(x, y) = \frac{\partial a(x,y)}{\partial x_j}$ and similarly for $\nabla_{x,k}\ a(x, y)$.

We define the mass density ρ and the momentum vector $\vec{p}$, by the relations

$$\rho = |u|^2, \qquad\qquad p_k = \Im(\bar{u}\nabla_k u).$$

It is well known, [11], that smooth solutions to the semilinear Schrödinger equation satisfy mass and momentum conservation. The local conservation of mass reads

$$\partial_t \rho + 2 div\vec{p} = \partial_t \rho + 2\nabla_j p^j = 0 \tag{4.10}$$

and the local momentum conservation is

$$\partial_t p^j + \nabla^k \Big(\delta_k^j \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big) + \sigma_k^j \Big) = 0 \tag{4.11}$$

where the symmetric tensor σ_{jk} is given by

$$\sigma_{jk} = 2\Re(\nabla_j u \nabla_k \overline{u}).$$

Notice that the term $\lambda\frac{p-1}{p+1}|u|^{p+1}$ is the only nonlinear term that appears in the expression. One can express the local conservation laws purely in terms of the mass density ρ and the momentum $\vec{p}$ if we write

$$\lambda\frac{p-1}{p+1}|u|^{p+1} = 2^{\frac{p+1}{2}}\lambda\frac{p-1}{p+1}\rho^{\frac{p+1}{2}}$$

and

$$\sigma_{jk} = 2\Re(\nabla_j u \nabla_k \overline{u}) = \frac{1}{\rho}(2p_j p_k + \frac{1}{2}\nabla_j \rho \nabla_k \rho),$$

but we will not use this formulation in these notes.

This is the main theorem of this section:

Theorem 4.1. [14, 17, 64, 72] *Consider $u \in C_t(\mathbb{R}; C_0^\infty(\mathbb{R}^n))$ a smooth and compactly supported solution to* (4.9) *with $u(x,0) = u(x) \in C_0^\infty(\mathbb{R}^n)$. Then for $n \geq 2$ we have that*

$$C\|D^{-\frac{n-3}{2}}(|u|^2)\|^2_{L_t^2 L_x^2} + (n-1)\lambda\frac{p-1}{p+1}\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^n\times\mathbb{R}_y^n} \frac{|u(y,t)|^2|u(x,t)|^{p+1}}{|x-y|}dxdydt$$
$$\leq \|u_0\|^2_{L^2}\sup_{t\in\mathbb{R}}|M_y(t)|,$$

where

$$M_y(t) = \int_{\mathbb{R}^n}\frac{x-y}{|x-y|}\cdot\Im\big(\overline{u}(x)\nabla u(x)\big)dx,$$

D^α is defined on the Fourier side as $\widehat{D^\alpha f}(\xi) = |\xi|^\alpha\widehat{u}(\xi)$ for any $\alpha \in \mathbb{R}$ and C is a positive constant that depends only on n, [66]*. For $n = 1$ the estimate is*

$$\|\partial_x(|u|^2)\|^2_{L_t^2 L_x^2} + \lambda\frac{p-1}{p+1}\|u\|^{p+3}_{L_t^{p+3}L_x^{p+3}} \leq \frac{1}{2}\|u_0\|^3_{L^2}\sup_{t\in\mathbb{R}}\|\partial_x u\|_{L^2}.$$

Remarks on Theorem 4.1.

1. By the Cauchy-Schwarz inequality it follows that for any $n \geq 2$,

$$\sup_{0,t}|M_y(t)| \lesssim \|u_0\|_{L^2}\sup_{t\in\mathbb{R}}\|\nabla u(t)\|_{L^2}.$$

A variant of Hardy's inequality gives

$$\sup_{0,t}|M_y(t)| \lesssim \sup_{t\in\mathbb{R}}\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}},$$

For details, see [34].

2. Concerning our main theorem, we note that both the integrated functions in the second term on the left hand side of the inequalities are positive. Thus when $\lambda > 0$, which corresponds to the defocusing case, and for H^1 data say, we obtain for $n \geq 2$:

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u_0\|_{L^2}^{\frac{3}{2}} \sup_{t\in\mathbb{R}} \|\nabla u(t)\|_{L^2}^{\frac{1}{4}} \lesssim M(u_0)^{\frac{3}{2}} E(u_0)^{\frac{1}{4}},$$

and for $n = 1$

$$\|\partial_x(|u|^2)\|_{L_t^2 L_x^2}^2 \lesssim \|u_0\|_{L^2}^{\frac{3}{2}} \sup_{t\in\mathbb{R}} \|\partial_x u(t)\|_{L^2}^{\frac{1}{2}} \lesssim M(u_0)^{\frac{3}{2}} E(u_0)^{\frac{1}{4}}.$$

These are easy consequences of the conservation laws of mass (4.3) and energy (4.2). They provide the global a priori estimates that are used in quantum scattering in the energy space, [34].

3. Analogous estimates hold for the case of the Hartree equation $iu_t + \Delta u = \lambda(|x|^{-\gamma} \star\ |u|^2)u$ when $0 < \gamma < n$, $n \geq 2$. For the details, see [42]. We should point out that for $0 < \gamma \leq 1$ scattering fails for the Hartree equation, [38], and thus the estimates given in [42] for $n \geq 2$ cover all the interesting cases.

4. Take $\lambda > 0$. The expression

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L_t^2 L_x^2},$$

for $n = 3$, provides an estimate for the $L_t^4 L_x^4$ norm of the solution. For $n = 2$ by Sobolev embedding one has that

$$\|u\|_{L_t^4 L_x^8}^2 = \||u|^2\|_{L_t^2 L_x^4} \lesssim \|D^{\frac{1}{2}}\left(|u|^2\right)\|_{L_t^2 L_x^2} \lesssim C_{M(u_0),E(u_0)}.$$

For $n \geq 4$ the power of the D operator is negative but some harmonic analysis and interpolation with the trivial inequality

$$\|D^{\frac{1}{2}} u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}$$

provides an estimate in a Strichartz norm. For the details see [72].

5. In the defocusing case all the estimates above give a priori information for the $\dot{H}^{\frac{1}{4}}$-critical Strichartz norm. We remind the reader that the $\dot{H}^s$-critical Strichartz norm is $\|u\|_{L_t^q L_x^r}$ where the pair (q, r) satisfies $\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s$. In principle the correlation of k particles will provide a priori information for the $\dot{H}^{\frac{1}{2k}}$ critical Strichartz norm. In 1d an estimate that provides a bound on the $\dot{H}^{\frac{1}{8}}$ critical Strichartz norm has been given in [16].

6. To make our presentation easier we considered smooth solutions of the NLS equation. To obtain the estimates in Theorem 4.1 for arbitrary H^1 functions we have to regularize the solutions and then take a limit. The process is described in [34].

7. A more general bilinear estimate can be proved if one correlates two different solutions (thus considering different density functions ρ_1 and ρ_2). Unfortunately, one can obtain useful estimates only for $n \geq 3$. The proof is based on the fact that $-\Delta^2|x|$ is a positive distribution only for $n \geq 3$. For details the reader can check [17]. Our proof shows that the diagonal case when $\rho_1 = \rho_2 = |u|^2$ provides useful monotonicity formulas in all dimensions.

4.1. **One particle Morawetz inequalities.**

Proof. We define the Morawetz action centered at zero by

$$M_0(t) = \int_{\mathbb{R}^n} \nabla a(x) \cdot \vec{p}(x)\ dx, \tag{4.12}$$

where the weight function $a(x) : \mathbb{R}^n \to \mathbb{R}$ is for the moment arbitrary. The minimal requirements on $a(x)$ call for the matrix of the second partial derivatives $\partial_j \partial_k a(x)$ to be positive definite. Throughout our paper we will take $a(x) = |x|$, but many estimates can be given with different weight functions, see for example [15] and [52]. If we differentiate the Morawetz action with respect to time we obtain:

$$\partial_t M_0(t) = \int_{\mathbb{R}^n} \nabla a(x) \cdot \partial_t \vec{p}(x)\ dx = \int_{\mathbb{R}^n} \nabla_j a(x) \partial_t p^j(x)\ dx$$

$$= \int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) \delta_k^j \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big) dx + 2\int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) \Re\big(\nabla^j \overline{u} \nabla_k u\big) dx,$$

where we use equation (4.11). We rewrite and name the equation as follows

$$\partial_t M_0(t) = \int_{\mathbb{R}^n} \Delta a(x) \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big) dx + 2\int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) \Re\big(\nabla^j \overline{u} \nabla_k u\big) dx. \tag{4.13}$$

Notice that for $a(x) = |x|$ the matrix $\nabla_j \nabla_k a(x)$ is positive definite and the same is true if we translate the weight function by any point $y \in \mathbb{R}^n$ and consider $\nabla_{x,j} \nabla^{x,k} a(x-y)$ for example. That is for any vector function on $\mathbb{R}^n$, $\{v_j(x)\}_{j=1}^n$, with values on $\mathbb{R}$ or $\mathbb{C}$ we have that

$$\int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) v^j(x) v_k(x) dx \geq 0.$$

To see this, observe that for $n \geq 2$ we have $\nabla_j a = \frac{x_j}{|x|}$ and $\nabla_j \nabla_k a = \frac{1}{|x|}\big(\delta_{kj} - \frac{x_j x_k}{|x|^2}\big)$. Summing over $j = k$ we obtain $\Delta a(x) = \frac{n-1}{|x|}$. Then

$$\nabla_j \nabla^k a(x) v^j(x) v_k(x) = \frac{1}{|x|}\big(\delta_j^k - \frac{x_j x^k}{|x|^2}\big) v^j(x) v_k(x) = \frac{1}{|x|}\Big(|\vec{v}(x)|^2 - \big(\frac{x \cdot \vec{v}(x)}{|x|}\big)^2\Big) \geq 0$$

by the Cauchy-Schwarz inequality. Notice that it does not matter if the vector function is real or complex valued for this inequality to be true. In dimension one (4.13) simplifies to

$$\partial_t M_0(t) = \int_{\mathbb{R}} a_{xx}(x)\big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1} + 2|u_x|^2\big) dx. \tag{4.14}$$

In this case for $a(x) = |x|$, we have that $a_{xx}(x) = 2\delta(x)$. Since the identity (4.13) does not change if we translate the weight function by $y \in \mathbb{R}^n$ we can define the

Morawetz action with center at $y \in \mathbb{R}^n$ by

$$M_y(t) = \int_{\mathbb{R}^n} \nabla a(x-y) \cdot \vec{p}(x)\ dx.$$

We can then obtain like before

$$\partial_t M_y(t) = \int_{\mathbb{R}^n} \Delta_x a(x-y)\big(-\frac{1}{2}\Delta\rho + \lambda\frac{p-1}{p+1}|u|^{p+1}\big)dx \tag{4.15}$$

$$+2\int_{\mathbb{R}^n} \big(\nabla_{x,j}\nabla^{x,k} a(x-y)\big)\Re\big(\nabla^{x,j}\overline{u}\nabla_{x,k}u\big)dx. \tag{4.16}$$

Recall that

$$\partial_t M_0 = \int_{\mathbb{R}^n} \Delta a(x)\left(\frac{\lambda(p-1)}{p+1}|u|^{p+1} - \frac{1}{2}\Delta\rho\right)dx + \int_{\mathbb{R}^n} (\partial_j\partial^k a(x))\sigma_k^j dx$$

for a general weight function $a(x)$.
If we pick $a(x) = |x|^2$, then $\Delta a(x) = 2n$ and $\partial_j\partial_k a(x) = 2\delta_{kj}$. Therefore

$$\begin{aligned}
\partial_t M &= \frac{2n\lambda(p-1)}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx + 2\int_{\mathbb{R}^n} |\nabla u|^2 dx \\
&= 8\left(\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx\right) - \frac{2\lambda}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1}dx \\
&= 8E(u(t)) - \frac{2\lambda}{p+1}\left(4-n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1}dx. \qquad (4.17)
\end{aligned}$$

Thus if we define the quantity

$$V(t) = \int_{\mathbb{R}^n} a(x)\rho(x)dx,$$

with $a(x) = |x|^2$, we have that

$$\partial_t V(t) = \int_{\mathbb{R}^n} a(x)\partial_t\rho(x)dx = -2\int_{\mathbb{R}^n} a(x)\ \nabla\cdot\vec{p}\ dx = 2M(t) \tag{4.18}$$

using integration by parts. Thus

$$\partial_t^2 V(t) = 16E(u(t)) - \frac{4\lambda}{p+1}\left(4-n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1}dx. \tag{4.19}$$

Another useful calculation is the following. Set

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx.$$

Then we have:

$$\begin{aligned}
K(t) &= \|xu\|_{L^2}^2 + 4t^2\|\nabla u\|_{L^2}^2 - 4t\int_{\mathbb{R}^n} x\cdot p\ dx + \frac{8t^2\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx \\
&= \int_{\mathbb{R}^n} a(x)\rho(x)dx + 8t^2E(u(t)) - 2t\int_{\mathbb{R}^n} \nabla a\cdot p\ dx \\
&= \int_{\mathbb{R}^n} a(x)\rho(x)dx + 8t^2E(u_0) - 2t\int_{\mathbb{R}^n} \nabla a\cdot p\ dx, \qquad (4.20)
\end{aligned}$$

with $a(x) = |x|^2$. However

$$\partial_t\int_{\mathbb{R}^n} a(x)\rho(x)dx = \int_{\mathbb{R}^n} \nabla a\cdot p\ dx$$

and thus

$$\partial_t K(t) = -2t \int_{\mathbb{R}^n} \partial_j a(x) \partial_t p^j dx + 16tE(u_0) = -2t\partial_t M(t) + 16tE(u_0).$$

If we use (4.17) we have that

$$\partial_t K(t) = \frac{4\lambda t}{p+1}\left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Notice that for $p = 1 + \frac{4}{n}$, the quantity $K(t)$ is conserved.

4.2. **Two particle Morawetz inequalities.** We now define the two-particle Morawetz action

$$M(t) = \int_{\mathbb{R}^n_y} |u(y)|^2 M_y(t)\ dy$$

and differentiate with respect to time. Using the identity above and the local conservation of mass law we obtain four terms

$$\begin{aligned}
\partial_t M(t) &= \int_{\mathbb{R}^n_y} |u(y)|^2 \partial_t M_y(t)\ dy + \int_{\mathbb{R}^n_y} \partial_t \rho(y) M_y(t)\ dy \\
&= \int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} |u(y)|^2 \Delta_x a(x-y)\big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big) dxdy \\
&\quad +2\int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} |u(y)|^2 \big(\nabla_{x,j}\nabla^{x,k} a(x-y)\big)\Re\big(\nabla^{x,j}\overline{u}\nabla_{x,k}u\big) dxdy \\
&\quad -2\int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} \nabla^{y,j} p_j(y) \nabla_{x,k} a(x-y) p^k(x) dxdy \\
&= I + II + III + 2\int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} p_j(y)\nabla^{y,j}\nabla_{x,k} a(x-y) p^k(x) dxdy
\end{aligned}$$

by integration by parts with respect to the $y-$variable. Since

$$\nabla^{y,j}\nabla_{x,k} a(x-y) = -\nabla^{x,j}\nabla_{x,k} a(x-y)$$

we obtain that

$$\begin{aligned}
\partial_t M(t) &= I + II + III - 2\int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} \nabla^{x,j}\nabla_{x,k} a(x-y) p_j(y) p^k(x) dxdy \qquad (4.21)\\
&= I + II + III + IV
\end{aligned}$$

where

$$I = \int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} |u(y)|^2 \Delta_x a(x-y)\big(-\frac{1}{2}\Delta\rho\big) dxdy,$$

$$II = \int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} |u(y)|^2 \Delta_x a(x-y)\big(\lambda \frac{p-1}{p+1}|u|^{p+1}\big) dxdy,$$

$$III = 2\int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} |u(y)|^2 \big(\nabla_{x,j}\nabla^{x,k} a(x-y)\big)\Re\big(\nabla^{x,j}\overline{u}\nabla_{x,k}u\big) dxdy,$$

$$IV = -2\int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} \nabla^{x,j}\nabla_{x,k} a(x-y) p_j(y) p^k(x) dxdy.$$

Claim: $III+IV\geq 0$. Assume the claim. Since $\Delta_x a(x-y)=\frac{n-1}{|x-y|}$ we have that

$$\partial_t M(t)\geq \frac{n-1}{2}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\frac{|u(y)|^2}{|x-y|}\big(-\Delta\rho\big)dxdy+(n-1)\lambda\frac{p-1}{p+1}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\frac{|u(y)|^2}{|x-y|}|u(x)|^{p+1}dxdy.$$

But recall that on one hand we have that $-\Delta=D^2$ and on the other that the distributional Fourier transform of $\frac{1}{|x|}$ for any $n\geq 2$ is $\frac{c}{|\xi|^{n-1}}$ where c is a positive constant depending only on n. Thus we can define

$$D^{-(n-1)}f(x)=c\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|}dy$$

and express the first term as

$$\frac{n-1}{2}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\frac{|u(y)|^2}{|x-y|}\big(-\Delta\rho\big)dxdy=c\frac{n-1}{2}<D^{-(n-1)}|u|^2,\,D^2|u|^2>=C\|D^{-\frac{n-3}{2}}|u|^2\|^2_{L^2_x}$$

by the usual properties of the Fourier transform for positive and real functions. Integrating from 0 to t we obtain the theorem in the case that $n\geq 2$.

Proof of the claim: Notice that

$$III+IV=2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x-y)\Big(|u(y)|^2\Re\big(\nabla^{x,j}\overline{u}(x)\nabla_{x,k}u(x)\big)-p^j(y)p_k(x)\Big)dxdy$$

$$=2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x-y)\Big(\frac{\rho(y)}{\rho(x)}\Re\big(u(x)(\nabla^{x,j}\overline{u}(x))\overline{u}(x)(\nabla_{x,k}u(x))\big)-p^j(y)p_k(x)\Big)dxdy.$$

Since

$$\nabla_{x,j}\nabla_{x,k}a(x-y)=\nabla_{y,j}\nabla_{y,k}a(y-x)$$

by exchanging the roles of x and y we obtain the same inequality and thus

$$III+IV=\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x-y)\Big(\frac{\rho(y)}{\rho(x)}\Re\big(u(x)(\nabla^{x,j}\overline{u}(x))\overline{u}(x)(\nabla_{x,k}u(x))\big)-p^j(y)p_k(x)$$

$$+\frac{\rho(x)}{\rho(y)}\Re\big(u(y)(\nabla^{y,j}\overline{u}(y))\overline{u}(y)(\nabla_{y,k}u(y))\big)-p^j(x)p_k(y)\Big)dxdy.$$

Now set $z_1=\overline{u}(x)\nabla_{x,k}u(x)$ and $z_2=\overline{u}(x)\nabla^{x,j}u(x)$ and apply the identity

$$\Re(z_1\bar{z}_2)=\Re(z_1)\Re(z_2)+\Im(z_1)\Im(z_2)$$

to obtain

$$\Re\big(u(x)(\nabla^{x,j}\overline{u}(x))\overline{u}(x)(\nabla_{x,k}u(x))\big)=\Re\big(\overline{u}(x)\nabla_{x,k}u(x)\big)\Re\big(\overline{u}(x)\nabla^{x,j}u(x)\big)$$

$$+\Im\big(\overline{u}(x)\nabla_{x,k}u(x)\big)\Im\big(\overline{u}(x)\nabla^{x,j}u(x)\big)=\frac{1}{4}\nabla_{x,k}\rho(x)\nabla^{x,j}\rho(x)+p_k(x)p^j(x)$$

and similarly

$$\Re\big(u(y)(\nabla^{y,j}\overline{u}(y))\overline{u}(y)(\nabla_{y,k}u(y))\big)=\frac{1}{4}\nabla_{y,k}\rho(y)\nabla^{y,j}\rho(y)+p_k(y)p^j(y).$$

Thus

$$III+IV=\frac{1}{4}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x-y)\frac{\rho(y)}{\rho(x)}\nabla_{x,k}\rho(x)\nabla^{x,j}\rho(x)dxdy$$

$$+\frac{1}{4}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{y,j}\nabla^{y,k}a(x-y)\frac{\rho(x)}{\rho(y)}\nabla_{y,k}\rho(y)\nabla^{y,j}\rho(y)dxdy$$

$$+\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x} \nabla_{y,j}\nabla^{y,k}a(x-y)\Big(\frac{\rho(y)}{\rho(x)}p_k(x)p^j(x)+\frac{\rho(x)}{\rho(y)}p_k(y)p^j(y)-p_k(x)p^j(y)-p_k(y)p^j(x)\Big)dxdy.$$

Since the matrix $\nabla_{x,j}\nabla^{x,k}a(x-y)=\nabla_{y,j}\nabla^{y,k}a(x-y)$ is positive definite, the first two integrals are positive. Thus,

$$III+IV\geq$$

$$\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x} \nabla_{x,j}\nabla^{x,k}a(x-y)\Big(\frac{\rho(y)}{\rho(x)}p_k(x)p^j(x)+\frac{\rho(x)}{\rho(y)}p_k(y)p^j(y)-p_k(x)p^j(y)-p_k(y)p^j(x)\Big)dxdy.$$

Now if we define the two point vector

$$J_k(x,y)=\sqrt{\frac{\rho(y)}{\rho(x)}}p_k(x)-\sqrt{\frac{\rho(x)}{\rho(y)}}p_k(y)$$

we obtain that

$$III+IV\geq\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x-y)J^j(x,y)J_k(x,y)dxdy\geq 0$$

and we are done.

The proof when $n=1$ is easier. First, an easy computation shows that if $a(x,y)=|x-y|$ then $\partial_{xx}a(x,y)=2\delta(x-y)$. In this case from (4.21) we obtain

$$\partial_t M(t)=\int_{\mathbb{R}_y\times\mathbb{R}_x}|u(y)|^2 2\delta(x-y)\big(-\frac{1}{2}\rho_{xx}\big)dxdy+2\int_{\mathbb{R}}|u(x)|^2\big(\lambda\frac{p-1}{p+1}|u(x)|^{p+1}\big)dx$$

$$+4\int_{\mathbb{R}}|u(x)|^2|u_x|^2dx-4\int_{\mathbb{R}}p^2(x)dx.$$

But

$$\int_{\mathbb{R}_y\times\mathbb{R}_x}|u(y)|^2 2\delta(x-y)\big(-\frac{1}{2}\rho_{xx}\big)dxdy=\int_{\mathbb{R}}\Big(\partial_x|u(x)|^2\Big)^2dx.$$

In addition a simple calculation shows that

$$|u(x)|^2|u_x|^2=\Big(\Re(\overline{u}u_x)\Big)^2+\Big(\Im(\overline{u}u_x)\Big)^2=\frac{1}{4}\Big(\partial_x|u|^2\Big)^2+p^2(x).$$

Thus

$$4|u(x)|^2|u_x|^2-4p^2(x)=\Big(\partial_x|u|^2\Big)^2$$

and the identity becomes

$$\partial_t M(t)=2\int_{\mathbb{R}}\Big(\partial_x|u|^2\Big)^2dx+2\int_{\mathbb{R}}|u(x)|^2\big(\lambda\frac{p-1}{p+1}|u(x)|^{p+1}\big)dx \tag{4.22}$$

which finishes the proof of the theorem. □

4.3. **Applications.** We now present a few applications of the decay estimates that were established above.

4.3.1. *Blow–up for the energy sub-critical and mass (super)–critical problem .* We show a criterion for blow-up for the energy subcritical and mass critical or supercritical

$$1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$$

focusing ($\lambda = 1$) problem which is due to Zakharov and Glassey. In our presentation we follow [36]. In addition we assume that our data have some decay (which will be specified below).

From the lwp theory we have a well-defined solution in $(0, T^\star)$ of the following initial value problem:

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2 dx), & \end{cases} \tag{4.23}$$

for any $1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}$ when $n \geq 3$, and for $1 + \frac{4}{n} \leq p < \infty$ when $n = 1, 2$.

Recall that for the variance, which was introduced as follows:

$$V(t) = \int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx,$$

we calculated that (see (4.18) and (4.19) and expressions leading to them):

$$\partial_t V(t) = 2M(t), \tag{4.24}$$

where

$$M(t) = \int_{\mathbb{R}^n} \vec{x} \cdot \vec{p}\, dx = \int_{\mathbb{R}^n} \vec{x} \cdot \Im(\bar{u} \nabla u)\ dx,$$

and

$$\partial_t^2 V(t) = 16E(u(t)) + \frac{4}{p+1}\left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx. \tag{4.25}$$

Hence (4.25) together with conservation of energy and the fact that $p \geq 1 + \frac{4}{n}$, implies:

$$\partial_t^2 V(t) \leq 16E(u_0),$$

which we can integrate twice to obtain:

$$\begin{aligned} V(t) &\leq 8t^2 E(u_0) + tV'(0) + V(0) \\ &= 8t^2 E(u_0) + 2tM(0) + V(0) \\ &= 8t^2 E(u_0) + 4t \int_{\mathbb{R}^n} \vec{x} \cdot \Im(\overline{u_0} \nabla u_0)\ dx + \|xu_0\|_{L^2}^2. \end{aligned} \tag{4.26}$$

Since

$$u_0 \in \Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2 dx),$$

the coefficients of the second degree polynomial in t on the right hand side of (4.26) are finite. Now if the initial data have negative energy, that is if

$$E(u_0) < 0,$$

the coefficient of t^2 is negative. On the other hand, for all times

$$V(t) = \int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx \geq 0.$$

Therefore $V(t)$ starts with a positive value $V(0)$ and at some finite time the second order polynomial $V(t)$ will cross the horizontal axis. Thus $T^\star$ is finite. By the blow–up alternative of the lwp theory this gives that

$$\lim_{t\to T^\star} \|u(t)\|_{H^1} = \infty,$$

if in addition to $u_0 \in H^1$, we have that $\|xu_0\|_{L^2} < \infty$ and $E(u_0) < 0$.

Remark 4.2. *We make a few comments:*

(1) *Note that the assumption $E(u_0) < 0$ is a sufficient condition for finite-time blow–up, but it is not necessary. One can actually prove that for any $E_0 > 0$ there exists u_0 with $E(u_0) = E_0$ and $T^\star < \infty$. For details consult* [11].
(2) *One can reasonably ask whether she can prove the same result for H^1 data? The authors in* [62] *prove such a result with the additional assumption of radial symmetry for any $n \geq 2$. For the L^2-critical case ($p = 1 + \frac{4}{n}$) the radial assumption is not needed. See the papers* [63, 35, 61] *for details.*
(3) *Many results have been devoted to the rate of the blow-up for the focusing problem. A variant of the local well-posedness theory provides the following result:*
If $u_0 \in H^1$ and $T^\star < \infty$, then there exists a $\delta > 0$ such that for all $0 \leq t < T^\star$ we have that

$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T^\star - t)^{\frac{1}{p-1} - \frac{n-2}{4}}}.$$

Note that the above gives a lower estimate but not an upper estimate. The authors in [56] *have provided an upper estimate for the L^2-critical case that is very close to the one above.*

4.3.2. *Global Well–Posedness for the L^2–critical problem.* We have seen that in the mass–critical case when $p = 1 + \frac{4}{n}$ the local existence time depends not only on the norm of the initial data but also on the profile. This prevents the use of the conservation of mass law in order to extend the solutions globally, even in the defocusing case ($\lambda = -1$). Here we discuss two interesting cases of global well–posedness under additional assumptions. The problem in full generality is developed in [20, 21, 22, 23].

Case 1: *Defocusing problem under the finite variance assumption* In the case when $\lambda < 0$, the conjecture was (for a long time) that $T^\star = \infty$. Although the conjecture is proven to be true in [20, 21, 22], in these notes we present a positive answer to an easier problem where we consider the corresponding problem for H^1 data (that can be large), but in addition we assume finiteness of the variance. This scenario can be analyzed using methods we developed so far and as such it fits well into our presentation.

Recall that

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx$$

is a conserved quantity for $p = 1 + \frac{4}{n}$. Thus

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx = \|xu_0\|_{L^2}^2.$$

We approximate the data with an H^1 sequence such that $u_{0,n} \to u_0$ in L^2 and have finite variance. The corresponding solutions satisfy $u_n \in C(\mathbb{R}, H^1(\mathbb{R}^n))$ and $xu_n \in C(\mathbb{R}, L^2(\mathbb{R}^n))$. The conservation law for $K(t)$ implies that

$$\frac{8t^2}{p+1}\int_{\mathbb{R}^n} |u_n|^{p+1}dx \leq C$$

and thus

$$\int_{\mathbb{R}^n} |u_n|^{2+\frac{4}{n}}dx \leq \frac{C}{t^2}$$

for all $t \in (0, T^\star)$. By continuous dependence this implies that

$$\int_{\mathbb{R}^n} |u(x,t)|^{2+\frac{4}{n}}dx \leq \frac{C}{t^2}$$

for a.a. $t \in (0, T^\star)$. Thus if $T^\star < \infty$ one can integrate the above quantity from any $t < T^\star$ to $T^\star$ and obtain that

$$\int_t^{T^\star}\int_{\mathbb{R}^n} |u(x,t)|^{2+\frac{4}{n}}dxdt < C.$$

Since on the other hand we have that

$$u \in L_t^{2+\frac{4}{n}}((0,t); L_x^{2+\frac{4}{n}})$$

we conclude

$$L_t^{2+\frac{4}{n}}((0,T^\star); L_x^{2+\frac{4}{n}}) < \infty.$$

But this contradicts the blow-up alternative for this problem and thus $T^\star = \infty$. Actually since the L^2 Strichartz norm $L_t^{2+\frac{4}{n}}L_t^{2+\frac{4}{n}}$ is bounded we also have scattering (more on that later).

Case 2: *Focusing problem.* Now let us derive a global well–posedness condition for the focusing equation

$$iu_t + \Delta u + |u|^{\frac{4}{n}}u = 0. \tag{4.27}$$

We have already seen that for small enough L^2 data the problem, focusing or defocusing, has global solutions. We have also mentioned the result in [23] that gives a sharp criterion for global existence for the focusing problem. Here we reproduce the result in [76] which states that if one assumes small L^2 data (but not arbitrarily small), which are, in addition, in H^1, global well–posedness follows by discovering the sharp constant of the Gagliardo–Nirenberg inequality.

More precisely since

$$\|u(t)\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \leq C\|\nabla u(t)\|_{L^2}^2\|u(t)\|_{L^2}^{\frac{4}{n}} = C\|\nabla u(t)\|_{L^2}^2\|u_0\|_{L^2}^{\frac{4}{n}},$$

one can easily see that the energy functional

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2dx - \frac{1}{2+\frac{4}{n}}\int |u(t)|^{2+\frac{4}{n}}dx$$

is bounded from below as follows

$$E(u(t)) = E(u_0) \geq \|\nabla u(t)\|_{L^2}^2 \left(\frac{1}{2} - C\|u_0\|_{L^2}^{\frac{4}{n}}\right). \tag{4.28}$$

Thus for $\|u_0\|_{L^2} < \eta$, η a fixed number, we have that

$$\|\nabla u(t)\|_{L^2} + \|u(t)\|_{L^2} \leq C_{M(u_0),E(u_0)} < \infty.$$

By the blow–up alternative of the H^1 theory we see that $T_{\max} = \infty$.

The question remains what is the optimal η. It was conjectured that, even with L^2–data, the optimal η is the mass of the ground state Q, which is the solution to the elliptic equation:

$$-Q + \Delta Q = |Q|^{\frac{4}{n}} Q,$$

that can be obtained by using the the ansatz $u(x,t) = e^{it}Q(x)$ in (4.27). It is shown that Q is unique, positive, spherically symmetric and very smooth (see [11] for exact references). Also Q satisfies certain identities (Pohozaev's identities) that can be obtained by multiplying the elliptic equation by $\bar{u}$ and $x \cdot \nabla u$ and take the real part respectively. In particular the identities imply that $E(Q) = 0$. In [76] Weinstein discovered that the mass of the ground state is related to the best constant of the Gagliardo–Nirenberg inequality. More precisely by minimizing the functional

$$J(u) = \frac{\|\nabla u(t)\|_{L^2}^2 \|u\|_{L^2}^{\frac{4}{n}}}{\|u\|_{L_x^{2+\frac{4}{n}}}^{2+\frac{4}{n}}},$$

Weinstein showed that the best constant of the Galgiardo–Nirenberg inequality

$$\frac{1}{2+\frac{4}{n}} \|u(t)\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \leq \frac{C}{2} \|\nabla u(t)\|_{L^2}^2 \|u(t)\|_{L^2}^{\frac{4}{n}},$$

is

$$C = \|Q\|_{L^2}^{-\frac{4}{n}}.$$

Hence we can revisit (4.28) to obtain

$$E(u_0) \geq \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 \left(1 - \frac{\|u_0\|_{L^2}^{\frac{4}{n}}}{\|Q\|_{L^2}^{\frac{4}{n}}}\right).$$

Therefore, if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, we have a global solution.

Moreover the condition is sharp in the sense that for any $\eta > \|Q\|_{L^2}$, there exists $u_0 \in H^1$ such that $\|u_0\|_{L^2} = \eta$, and $u(t)$ blows-up in finite time. To see that, set

$$\gamma = \frac{\eta}{\|Q\|L^2} > 1,$$

and consider $u_0 = \gamma Q$. Then $\|u_0\|_{L^2} = \eta$ and

$$E(u_0) = \gamma^{2+\frac{4}{n}} E(Q) - \frac{\gamma^{2+\frac{4}{n}} - \gamma^2}{2} \|\nabla Q\|_{L^2}^2 = -\frac{\gamma^{2+\frac{4}{n}} - \gamma^2}{2} \|\nabla Q\|_{L^2}^2 < 0.$$

Since $u_0 = \gamma Q \in \Sigma$ and $E(u_0) < 0$, by the Zakharov–Glassey argument we have blow-up in finite time.

Remark 4.3. *As consequence of the pseudo-conformal transformation*

$$u(x,t) \to (1-t)^{-\frac{n}{2}} e^{-\frac{i|x|^2}{4(1-t)}} u(\frac{t}{1-t}, \frac{x}{1-t}),$$

we actually have blow–up even for $\eta = \|Q\|_{L^2}$. We cite [11] *for the details. It is interesting that the blow–up rate is $\frac{1}{t}$ and thus at least in the L^2–critical case the lower estimate we gave is not optimal for all blow–up solutions.*

4.3.3. *Quantum scattering in the energy space.* Consider the defocusing L^2–super–critical problem

$$\begin{cases} iu_t + \Delta u - |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n), \end{cases} \tag{4.29}$$

for any $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$.
We define the set of initial values u_0 which have a scattering state at $+\infty$ (by time reversibility all the statements are equivalent at $-\infty$):

$$\mathcal{R}_+ = (u_0 \in H^1 : T^\star = \infty, \quad u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t) \textit{ exists }). \tag{4.30}$$

Now define the operator

$$U : \mathcal{R}_+ \to H^1.$$

This operator sends u_0 to the scattering state u_+. If this operator is injective then we can define the wave operator

$$\Omega_+ = U^{-1} : U(\mathcal{R}_+) \to \mathcal{R}_+$$

which sends the scattering state u_+ to u_0. Thus the first problem of scattering is the existence of wave operator:
• *Existence of wave operators. For each u_+ there exists unique $u_0 \in H^1$ such that $u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$.*
If the wave operator is also surjective we say that we have asymptotic completeness (thus in this case the wave operator is invertible):
• *Asymptotic completeness. For every $u_0 \in H^1$ there exists u_+ such that $u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$.*
Both statements make rigorous the idea that we have scattering if, as time goes to infinity, the nonlinear solution of the NLS behaves like the solution of the linear equation.
Using the decay estimates we have established we can solve the scattering problem for every $p > 1 + \frac{4}{n}$. Well defined wave operators for this range of p is easy and it is almost a byproduct of the local theory. But asymptotic completeness is hard. In dimensions $n \geq 3$ this was proved in [33] and for $n = 1, 2$ in [59, 60]. The proofs are complicated since they were achieved before the interaction Morawetz estimates. Using the interaction Morawetz estimates we can prove the scattering properties in two simple steps. To make the presentation clear we will only show the $n = 3$ case with the cubic nonlinearity. But keep in mind that the interaction Morawetz estimates give global a priori control on quantities of the form

$$\|u\|_{L_t^q L_x^r} \leq C_{M(u_0),E(u_0)},$$

for certain q and r in all dimensions. It turns out that in the L^2–supercritical case this is enough to give scattering for any $p > 1 + \frac{4}{n}$ and n. Finally for completeness we also outline the wave operator question.

Theorem 4.4. *For every $u_+ \in H^1(\mathbb{R}^3)$ there exists unique $u_0 \in H^1(\mathbb{R}^3)$ such that the maximal solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ of $iu_t + \Delta u = |u|^2 u$, satisfies*

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\|_{H^1(\mathbb{R}^3)} = 0.$$

Proof: For $u_+ \in H^1$ define the map

$$\mathcal{A}(u)(t) = e^{it\Delta}u_+ + i\int_t^\infty e^{i(t-s)\Delta}(|u|^2u)(s)ds.$$

What is the motivation behind this map? Recall that

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}(|u|^2u)(s)ds,$$

$$e^{-it\Delta}u(t) = u_0 - i\int_0^t e^{-is\Delta}(|u|^2u)(s)ds. \tag{4.31}$$

If the problem scatters we have that $\lim_{t\to\infty}\|e^{-it\Delta}u(t) - u_+\|_{H^1} = 0$ and thus

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^2u)(s)ds \tag{4.32}$$

in H^1 sense. Now subtracting (4.32) from (4.31) we have that

$$u(t) = e^{it\Delta}u_+ + i\int_t^\infty e^{i(t-s)\Delta}(|u|^2u)(s)ds.$$

By Strichartz estimates we have that

$$\|e^{it\Delta}u_+\|_{L_t^qW_x^{1,r}} \lesssim \|u_+\|_{H^1} < \infty.$$

By the monotone convergence theorem there exists $T = T(u_+)$ large enough such that for $q < \infty$ we have

$$\|e^{it\Delta}u_+\|_{L_t^qW_x^{1,r}} \lesssim \epsilon.$$

The trick here is to use the smallness assumption to iterate the map in the interval (T,∞). But our local theory was performed in the norms

$$\|u\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$

where

$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible}\|u\|_{L^q_{t\in I}L^r_x}.$$

But this norms contain L_t^∞. So momentarily we will go to the smaller space

$$X = L_t^5L_x^5 \cap L_t^{\frac{10}{3}}W_x^{1,\frac{10}{3}}.$$

For this norm we also have that for large T

$$\|e^{it\Delta}u_+\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

Furthermore Strichartz estimates show that

$$\|\mathcal{A}(u)\|_{X_{[T,\infty)}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

The main step here is Sobolev embedding

$$\|f\|_{L_t^5L_x^5} \lesssim \|f\|_{L_t^5W_x^{1,\frac{30}{11}}}$$

where the pair $(5, \frac{30}{11})$ is Strichartz admissible. The details are as follows: Notice that the dual pair of $(\frac{10}{3}, \frac{10}{3})$ is $(\frac{10}{7}, \frac{10}{7})$.

$$\|u\|_{L_t^5 L_x^5} \lesssim \|e^{it\Delta}u_+\|_{L_t^5 L_x^5} + \|\int_T^\infty e^{i(t-s)\Delta}\Big(|u|^2 u(s)\Big)ds\|_{L_t^5 L_x^5}$$

$$\lesssim \|e^{it\Delta}u_+\|_{L_t^5 W_x^{1\frac{30}{13}}} + \|\int_T^\infty e^{i(t-s)\Delta}\Big(|u|^2 u(s)\Big)ds\|_{L_t^5 W_x^{1,\frac{30}{13}}}$$

$$\lesssim \epsilon + \|u^3\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + \|(\nabla u)u^2\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}}$$

$$\lesssim \epsilon + \|u\|^2_{L_t^5 L_x^5}\|u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} + \|u\|^2_{L_t^5 L_x^5}\|\nabla u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}}$$

$$\lesssim \epsilon + \|u\|^2_{L_t^5 L_x^5}\|u\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

Similarly we derive

$$\|\nabla u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^2_{L_t^5 L_x^5}\|\nabla u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

and

$$\|u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^2_{L_t^5 L_x^5}\|u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

Thus for T large enough we have that

$$\|u\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

More precisely to obtain the last claim one has to estimate $\|\mathcal{A}\|_X$, $\|\mathcal{A}(u) - \mathcal{A}(v)\|_X$ and prove that the map $\mathcal{A}$ is a contraction. Thanks to the ϵ we derive simultaneously this property along with the estimate

$$\|u\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

It remains to show that the solution is in $C([T,\infty); H^1(\mathbb{R}^3))$. But by Strichartz again and using any admissible pair we have

$$\|u\|_{L^q_{t\in[T,\infty)} W_x^{1,r}} \lesssim \|u_+\|_{H^1} + \|u\|^3_{X_{[T,\infty)}} \lesssim \|u_+\|_{H^1}.$$

In particular $\psi = u(T) \in H^1$ and we have a strong H^1 solution of the equation with initial data $u(T) = \psi$. But we know that the solutions of this equation are global and thus $u(0)$ is well-defined. Finally

$$e^{-it\Delta}u(t) - u_+ = i\int_t^\infty e^{-is\Delta}(|u|^2 u)(s)ds,$$

$$\nabla\Big(e^{-it\Delta}u(t) - u_+\Big) = i\int_t^\infty e^{-is\Delta}\Big(\nabla(|u|^2 u)\Big)(s)ds,$$

$$\|e^{-it\Delta}u(t) - u_+\|_{H^1} \lesssim \|\nabla u\|_{L^{\frac{10}{3}}_{[t,\infty)} L_x^{\frac{10}{3}}}\|u\|^2_{L^5_{[t,\infty)} L_x^5} \lesssim \|u\|^3_{X_{[T,\infty)}}$$

But for T large enough we have that $\|u\|_{X_{[T,\infty)}} \lesssim \epsilon$ and thus

$$\lim_{t\to\infty}\|e^{-it\Delta}u(t) - u_+\|_{H^1} = 0.$$

Therefore $u(0) = u_0 \in H^1$ satisfies the assumptions of the theorem. We end with asymptotic completeness.

Theorem 4.5. *If $u_0 \in H^1(\mathbb{R}^3)$ and if $u \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ where u is the solution of $iu_t + \Delta u = |u|^2 u$, then there exists u_+ such that*

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\| = 0.$$

The proof is based on a simple proposition assuming the interaction Morawetz estimates. This was the hardest part in the earlier proofs of quantum scattering.

Proposition 4.6. *Let u be a global H^1 solution of the cubic defocusing equation on $\mathbb{R}^3$. Then*

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \le C.$$

Proof: We know that $\|u\|_{L_t^4 L_x^4} \le C$ for energy solutions. Thus we can pick ϵ small to be determined later and a finite number of intervals $\{I_k\}_{k=1,2,\dots,M}$, with $M < \infty$ such that

$$\|u\|_{L^4_{t\in I_k} L^4_x} \le \epsilon$$

for all k. If we apply the Strichartz estimates on each I_k we obtain for some $\alpha < 1$

$$\|u\|_{\mathcal{S}^1(I_k)} \lesssim \|u(t_{k-1})\|_{H^1} + \|u\|^{2\alpha}_{L^4_{t\in I_k} L^4_x} \|u\|^{3-2\alpha}_{\mathcal{S}^1(I_k)}, \tag{4.33}$$

$$\|u\|_{\mathcal{S}^1(I_k)} \lesssim \|u(t_{k-1})\|_{H^1} + \epsilon^{2\alpha} \|u\|^{3-2\alpha}_{\mathcal{S}^1(I_k)}.$$

We can pick ϵ so small such that

$$\|u\|_{\mathcal{S}^1(I_k)} \le K.$$

Since the number of intervals are finite and the conclusion can be made for all $I_k's$ the proposition follows.

Remarks. 1. Where do we use the condition $p > 1 + \frac{4}{n}$? This is a delicate matter. It is not hard to see that the interaction Morawetz estimates are global estimates of Strichartz type but are not L^2 scale invariant. If one inspects the right hand side of the interaction inequalities, a simple scaling argument shows that these are $H^{\frac{1}{4}}$ invariant estimates. Thus only in the case that $p > 1 + \frac{4}{n}$ we can take advantage of an non L^2 estimate such as $L_t^4 L_x^4$. This is the heart of the matter in proving (4.33). In the case that $p = 1 + \frac{4}{n}$ we need to have a global L^2 Strichartz estimate like $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}$ in dimensions 3. Estimates of this sort can never come from Morawetz estimates due to scaling.
2. Notice that the proposition gives a global decay estimate for the nonlinear solution.

Let's finish the proof of asymptotic completeness. Note that

$$e^{-it\Delta}u(t) = u_0 - i\int_0^t e^{-is\Delta}(|u|^2u)(s)ds,$$

$$e^{-i\tau\Delta}u(\tau) = u_0 - i\int_0^\tau e^{-is\Delta}(|u|^2u)(s)ds.$$

Thus

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} = \|u(t) - e^{i(t-\tau)\Delta}u(\tau)\|_{H^1} \lesssim \|u\|^3_{\mathcal{S}^1_{(t,\tau)}} \le C$$

again by Strichartz estimates. Thus as $t, \tau \to \infty$ we have that

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} \to 0.$$

By completeness of H^1 there exists $u_+ \in H^1$ such that $e^{-it\Delta}u(t) \to u_+$ in H^1 as $t \to \infty$. In particular in H^1 we have

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^2u)(s)ds$$

and thus

$$\|e^{-it\Delta}u(t) - u_+\|_{H^1} \lesssim \|u\|^3_{\mathcal{S}^1_{(t,\infty)}}.$$

As $t \to \infty$ the conclusion follows.

More remarks. What about energy scattering for $p \leq 1 + \frac{4}{n}$. The critical case has been solved in [20, 21, 22]. For $p < 1 + \frac{4}{n}$ the problem is completely open. We have already mentioned that scattering makes rigorous the intuition that as time increases, for a defocusing problem, the nonlinearity $|u|^{p-1}u$ becomes negligible. From this observation one expects that the bigger the power of p the better chance the solution has to scatter. Thus the question: Is there any threshold p_0 with $1 < p_0 \leq 1 + \frac{4}{n}$ such that energy scattering does fail? The answer is yes and $p_0 = 1 + \frac{2}{n}$. This is in [68] for higher dimensions and in [3] for dimension one. More precisely using the pseudo-conformal conservation law and decay estimates that we discuss later in the notes, they showed that for any $1 < p \leq p_0$, $U(-t)u(t)$ doesn't converge even in L^2. Thus the wave operators cannot exist in any reasonable set. The problem remains open for

$$1 + \frac{2}{n} < p < 1 + \frac{4}{n}$$

and for general energy data. For partial results see [42] and the references therein.

4.3.4. *Quantum scattering in the Σ space.* If we are willing to abandon the energy space can we improve scattering in the range $1 + \frac{2}{n} < p < 1 + \frac{4}{n}$? Recall that

$$\Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2dx).$$

We will not go into the details but a few comments can clarify the situation. Exactly like the energy case it is enough to prove that

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C.$$

How one can obtain this estimate for different values of p? First recall that for

$$K(t) = \|(x + 2it\nabla)u\|^2_{L^2} + \frac{8t^2}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx$$

we have that

$$K(t) - K(0) = \int_0^t \theta(s)ds,$$

where

$$\theta(t) = \frac{4t}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n}|u|^{p+1}dx.$$

Using this quantity and a simple analysis one can obtain the following proposition:

Proposition 4.7. *Consider the defocusing NLS*

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n). \end{cases} \tag{4.34}$$

for any $1 < p < 1 + \frac{4}{n-2}$, $n \geq 3$ *(*$1 < p < \infty$ *for* $n = 1, 2$*). If in addition* $\|xu_0\|_{L^2} < \infty$ *and*

$$u \in C_t^0(\mathbb{R}; H^1(\mathbb{R}^n))$$

solves (4.34)*, then we have:*
i) If $p > 1 + \frac{4}{n}$ *then for any* $2 \leq r \leq \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$*,* $2 \leq r < \infty$ *if* $n = 2$*)*

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})}$$

for all $t \in \mathbb{R}^n$*.*
ii) If $p < 1 + \frac{4}{n}$ *then for any* $2 \leq r \leq \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$*,* $2 \leq r < \infty$ *if* $n = 2$*)*

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})(1-\theta(r))}$$

where

$$\theta(r) = \begin{cases} 0 & \text{if } 2 \leq r \leq p+1 \\ \frac{[r-(p+1)][4-n(p-1)]}{(r-2)[(n+2)-p(n-1)]} & \text{if } r > p+1. \end{cases}$$

Remarks. 1. Notice that for $p \geq 1 + \frac{4}{n}$ the decay is as strong as the linear equation. Recall here the basic $L^1 - L^\infty$ estimate of the linear problem and its interpolation with Plancherel's theorem.
2. Using these estimates and the standard theory we have developed one can prove that global solutions defined in the Σ space obey

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C,$$

for any

$$1 + \frac{2 - n + \sqrt{n^2 + 12n + 4}}{2n} < p < 1 + \frac{4}{n-2}.$$

The existence of wave operators and asymptotic completeness follows easily. Of course

$$1 + \frac{2}{n} < 1 + \frac{2 - n + \sqrt{n^2 + 12n + 4}}{2n} < 1 + \frac{4}{n}.$$

3. The existence of the wave operators can go below the above threshold in all dimensions. Indeed one can cover the full range $p > 1 + \frac{2}{n}$. The subject is rather technical and we refer to [11] for more details.

5. The Korteweg de Vries equation.

5.1. **The Bona-Smith method.** In this section we prove existence and uniqueness of solutions for the KdV equation. In addition we obtain the continuity of solution with respect to the initial data. We start with the case of smooth solutions. We follow the proof in [4]. This method is independent of the dispersion relation of the equation and can be applied to a large class of nonlinear evolution PDE. The hardest part of this process is to establish the property of continuity with respect to the initial data.

More precisely consider the initial value problem:

$$u_t + u_{xxx} + uu_x = 0$$

with initial data $u(0,x) = u_0(x) \in H^s(\mathbb{R})$ with s being a sufficiently large integer. In this section, we say u is a classical solution of KdV in H^s if

$$u \in C([-\delta,\delta]; H^s) \cap C^1([-\delta,\delta]; H^{s-3})$$

and if u satisfies KdV for each x and t.

We start with the following energy inequality: if u is a smooth solution of KdV, then there exists $T_0 = T_0(\|u_0\|_{H^s})$ such that on $[0, T_0]$,

$$\|u\|_{H^s} \leq 2\|u_0\|_{H^s}.$$

Indeed,

$$\begin{aligned}\partial_t \|\partial_x^s u\|_{L^2}^2 &= 2\int \partial_x^s u_t \partial_x^s u dx \\ &= -2\int \partial_x^{s+3} u \partial_x^s u dx - 2\int \partial_x^s (uu_x) \partial_x^s u dx.\end{aligned}$$

The first term is zero, the highest order contribution of the second term is

$$-2\int u \partial_x^{s+1} u \partial_x^s u dx = \int u_x (\partial_x^s u)^2 dx.$$

Thus, we obtain for $s > 3/2$

$$\partial_t \|u\|_{H^s}^2 \lesssim \|u_x\|_{L^\infty} \|u\|_{H^s}^2 \lesssim \|u\|_{H^s}^3.$$

Integrating in time implies that

$$\|u(T)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 + \int_0^T \|u(\tau)\|_{H^s}^3 d\tau.$$

Let $T_0 = \inf\{T : \|u\|_{H^s} \geq 2\|u_0\|_{H^s}\}$. Then on $[0, T_0]$, we have

$$\|u(T)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 + 8T_0 \|u_0\|_{H^s}^3.$$

This implies that $T_0 \geq (8\|u_0\|_{H^s})^{-1}$.

Remark 5.1. *We also note that the above argument implies via Gronwall's inequality that*

$$\|u\|_{H^s} \leq C\|u_0\|_{H^s} \exp\Big(C\int_0^t \|u_x\|_{L^\infty} dt'\Big),$$

where C is an absolute constant. The advantage of this inequality is that the time that it is valid depends only on the lower index Sobolev norm (H^2 is enough, available by the energy inequality) of the initial data, whereas the a priori energy bound depends on the H^s norm.

To prove the existence and uniqueness of solutions we use parabolic regularization and consider

$$u_t + \epsilon u_{xxxx} + u_{xxx} + uu_x = 0.$$

The energy inequality above remains intact since the contribution of the parabolic term is negative. Local well-posedness of this equation is proved by running a contraction argument in the space

$$X_T = \{u \in C([0,T]; H^s) : u(0,x) = u_0(x), \sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}\},$$

for the operator

$$\Gamma u = e^{-\epsilon t\partial_x^4}u_0 - \int_0^t e^{-\epsilon(t-\tau)\partial_x^4}(u_{xxx} + uu_x)ds.$$

We have the following inequalities for the heat kernel:

$$\begin{aligned}
\|e^{-\epsilon t\partial_x^4}u\|_{H^s} &\leq \|u\|_{H^s} \\
\|e^{-\epsilon t\partial_x^4}\partial_x^3 u\|_{H^s} &\lesssim \frac{1}{\epsilon^{3/4}t^{3/4}}\|u\|_{H^s} \\
\|e^{-\epsilon t\partial_x^4}(uu_x)\|_{H^s} \lesssim \frac{1}{\epsilon^{1/4}t^{1/4}}\|u^2\|_{H^s} &\lesssim \frac{1}{\epsilon^{1/4}t^{1/4}}\|u\|_{H^s}^2.
\end{aligned}$$

The first one follows from the boundedness of the multiplier $e^{-\epsilon t\xi^4}$. The second follows by the inequality

$$\left|\xi^3 e^{-\epsilon t\xi^4}\right| \lesssim \frac{1}{\epsilon^{3/4}t^{3/4}}.$$

The third one follows similarly using the algebra property of Sobolev spaces.

Using these inequalities for Γ, we obtain

$$\begin{aligned}
\|\Gamma u(T)\|_{H^s} &\leq \|u_0\|_{H^s} + \int_0^T \frac{1}{\epsilon^{3/4}(T-\tau)^{3/4}}\|u\|_{H^s}d\tau + \int_0^T \frac{1}{\epsilon^{1/4}(T-\tau)^{1/4}}\|u\|_{H^s}^2 d\tau \\
&\leq \|u_0\|_{H^s} + C\epsilon^{-3/4}T^{1/4}\|u_0\|_{H^s} + C\epsilon^{-1/4}T^{3/4}\|u_0\|_{H^s}^2 \\
&\leq 2\|u_0\|_{H^s},
\end{aligned}$$

if $T \leq T_1(\epsilon, \|u_0\|_{H^s})$. Therefore, iterating this local result using the energy inequality we obtain a solution, u^ϵ, valid in the time interval $[0, T_0]$. Also note that, using the equation, we have $u^\epsilon \in C^1([0,T_0], H^{s-4})$. From now on we will denote T_0 by T.

Now we need to prove that u^ϵ converges to a solution of KdV as ϵ tends to zero. To do this we prove that u^ϵ is Cauchy in $C([0,T]; L^2)$. Take $0 < \epsilon < \epsilon'$ and consider the corresponding solutions. Using the equation for ϵ and ϵ', we have

$$\begin{aligned}
\partial_t\|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 &= -2\epsilon'\int(u^\epsilon - u^{\epsilon'})\partial_x^4(u^\epsilon - u^{\epsilon'}) - 2(\epsilon-\epsilon')\int(u^\epsilon - u^{\epsilon'})\partial_x^4 u^\epsilon \\
&- \int(u^\epsilon - u^{\epsilon'})\partial_x^3(u^\epsilon - u^{\epsilon'}) - \frac{1}{2}\int(u^\epsilon - u^{\epsilon'})\partial_x[(u^\epsilon - u^{\epsilon'})(u^\epsilon + u^{\epsilon'})] \\
&= -2\epsilon'\int(u^\epsilon - u^{\epsilon'})\partial_x^4(u^\epsilon - u^{\epsilon'}) - 2(\epsilon-\epsilon')\int(u^\epsilon - u^{\epsilon'})\partial_x^4 u^\epsilon \\
&- \frac{1}{4}\int(u^\epsilon - u^{\epsilon'})^2\partial_x(u^\epsilon + u^{\epsilon'}) \\
&\leq -2(\epsilon-\epsilon')\int(u^\epsilon - u^{\epsilon'})\partial_x^4 u^\epsilon - \frac{1}{4}\int(u^\epsilon - u^{\epsilon'})^2\partial_x(u^\epsilon + u^{\epsilon'}).
\end{aligned}$$

The second equality follows by noting that the third integral is zero. The last inequality follows by the inequality

$$-2\epsilon'\int(u^\epsilon - u^{\epsilon'})\partial_x^4(u^\epsilon - u^{\epsilon'}) = -2\epsilon'\int\left[\partial_x^2(u^\epsilon - u^{\epsilon'})\right]^2 \leq 0.$$

We estimate the remaining terms by Cauchy Schwarz and Sobolev embedding (for $\|\partial_x u\|_{L^\infty}$) to obtain

$$\partial_t \|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 \lesssim |\epsilon - \epsilon'| \|u^\epsilon - u^{\epsilon'}\|_{L^2} \|u^\epsilon\|_{H^4} + \|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 (\|u^\epsilon\|_{H^4} + \|u^{\epsilon'}\|_{H^4}).$$

This implies that

$$\partial_t \|u^\epsilon - u^{\epsilon'}\|_{L^2} \lesssim |\epsilon - \epsilon'| \|u^\epsilon\|_{H^4} + \|u^\epsilon - u^{\epsilon'}\|_{L^2} (\|u^\epsilon\|_{H^4} + \|u^{\epsilon'}\|_{H^4}).$$

Integrating from 0 to T for small T and using the apriori bound $\|u^{\epsilon'}\|_{H^4} \lesssim \|u_0\|_{H^4}$, we obtain

$$\sup_{[0,T]} \|u^\epsilon - u^{\epsilon'}\|_{L^2} \lesssim |\epsilon - \epsilon'|.$$

Therefore u^ϵ is Cauchy in $C([0,T]; L^2)$. By interpolation with $L^\infty H^s$, u^ϵ is also Cauchy in $C([0,T]; H^r)$ for any $r \in [0, s)$. Moreover using the equation, we conclude that $\partial_t u^\epsilon$ is Cauchy in $C([0,T]; H^{r-4})$. Therefore, the limiting function $u \in C([0,T]; H^r) \cap C^1([0,T]; H^{r-4})$ solves KdV (by taking pointwise limits). Also note that since u^ϵ is bounded in H^s and converges to u in L^2, $u \in L^\infty([0,T]; H^s)$.

We now continue with uniqueness. Consider KdV with initial data u_0 and v_0 in H^s. Let u and v be the corresponding solutions valid in a common time interval $[0,T]$. By using the equation as above we obtain

$$\begin{aligned}\partial_t \|u - v\|_{L^2}^2 &= -2 \int (u-v)\partial_x^3 (u-v) - \int (u-v)\partial_x (u^2 - v^2) \\ &\lesssim \|u - v\|_{L^2}^2 (\|u\|_{H^s} + \|v\|_{H^s}).\end{aligned}$$

Therefore by Gronwall we obtain

$$\|u - v\|_{L^2} \lesssim \|u_0 - v_0\|_{L^2} \tag{5.1}$$

on $[0,T]$. This implies uniqueness.

It remains to prove the continuous dependence on initial data and that $u \in C([0,T]; H^s)$ (this implies that $u \in C^1([0,T]; H^{s-3})$). To do this regularize the initial data as follows:

$$u_0^\delta := u_0 * \varphi_\delta.$$

Here φ is a Schwartz function with mean 1 satisfying $\partial_\xi^k \widehat{\varphi}(0) = 0$ for all $k > 0$ (take $\widehat{\varphi}$ constant 1 in a neighborhood of the origin), and $\varphi_\delta(x) = \frac{1}{\delta}\varphi(\frac{x}{\delta})$. Since φ_δ is an approximate identity, u_0^δ converges to u_0 in H^s. Note also that $\|u_0^\delta\|_{H^s} \lesssim \|u_0\|_{H^s}$ where the implicit constant is independent of δ. Let u^δ be the solution with initial data u_0^δ on $[0,T]$ (coming from the parabolic regularization as the limit of smooth ϵ-solutions). Note that u^δ is smooth and satisfies KdV.

We will need the following lemma

Lemma 5.2. *Consider the solutions u^δ constructed above on $[0,T]$. We claim that*
i) $\|u^\delta\|_{H^{s+1}} \lesssim 1/\delta$,
ii) u^δ *converges to* u *in* $L^\infty_{[0,T]} H^s$,
iii) $\|u^\delta - v^\delta\|_{H^s} \lesssim e^{CT/\delta} \|u_0 - v_0\|_{H^s}$,
iv) Assume that $u_{n,0}$ converges to u_0 in H^s. Let u_n and u_n^δ be the solutions corresponding to the initial data $u_{n,0}$ and $u_{n,0}^\delta$, respectively. Then

$$\sup_n \|u_n^\delta - u_n\|_{H^s} \to 0 \qquad \text{as } \delta \to 0.$$

The bound i) and interpolation imply that

$$u^\delta \in C([0,T];L^2) \cap L^\infty_{[0,T]} H^{s+1} \subset C([0,T];H^s).$$

This implies that $u \in C([0,T];H^s)$, since by ii) u^δ converges to u uniformly on $[0,T]$.

The lemma also implies continuous dependence on initial data as follows. Assume that $u_{n,0}$ converges to u_0 in H^s. Construct the regularized solutions as in the lemma. Using triangle inequality and iii), we have (for $t \in [0,T]$)

$$\begin{aligned}\|u - u_n\|_{H^s} &\le \|u - u^\delta\|_{H^s} + \|u^\delta - u_n^\delta\|_{H^s} + \|u_n^\delta - u_n\|_{H^s} \\ &\lesssim \|u - u^\delta\|_{H^s} + e^{CT/\delta}\|u_0 - u_{n,0}\|_{H^s} + \sup_j \|u_j^\delta - u_j\|_{H^s}.\end{aligned}$$

Given $\epsilon > 0$, fix δ_0 sufficiently small so that in light of ii) and iv) we have

$$\|u - u_n\|_{H^s} \lesssim \epsilon + e^{CT/\delta_0}\|u_0 - u_{n,0}\|_{H^s} + \epsilon.$$

Taking n to ∞ finishes the proof. It remains to prove the lemma.

Proof of Lemma 5.2. i) We first recall that on $[0,T]$

$$\|u^\delta\|_{H^{s+1}} \lesssim \|u_0^\delta\|_{H^{s+1}} \exp\Big(C\int_0^t \|u_x^\delta\|_\infty dt'\Big) \lesssim \|u_0^\delta\|_{H^{s+1}} \exp\big(CT\|u_0^\delta\|_{H^s}\big).$$

Therefore, it suffices to prove i) at time 0. Indeed,

$$\|u_0^\delta\|_{H^{s+1}} = \|\langle\xi\rangle\widehat{\varphi}(\delta\xi)\langle\xi\rangle^s\widehat{u_0}(\xi)\|_{L^2} \lesssim \|\langle\xi\rangle\widehat{\varphi}(\delta\xi)\|_{L^\infty}\|u_0\|_{H^s} \lesssim 1/\delta.$$

ii) We first prove that for $0 < \delta' < \delta$, we have

$$\|u^\delta - u^{\delta'}\|_{L^2} = o(\delta^s).$$

By (5.1), it suffices to prove this at time zero. We have

$$\|u_0^\delta - u_0^{\delta'}\|_{L^2}^2 = \int |\widehat{\varphi}(\delta\xi) - \widehat{\varphi}(\delta'\xi)|^2\langle\xi\rangle^{-2s}|\widehat{u_0}(\xi)|^2\langle\xi\rangle^{2s}d\xi.$$

By Taylor expansion and the fact that derivatives of $\widehat{\varphi}$ vanishes at zero, we have

$$\widehat{\varphi}(\delta\xi) = 1 + O(\delta^s\xi^s \sup_{[0,\delta\xi]} |\partial^s\widehat{\varphi}|).$$

Thus, we have

$$\|u_0^\delta - u_0^{\delta'}\|_{L^2}^2 \lesssim \delta^{2s}\int \|\partial^s\widehat{\varphi}\|_{L^\infty([0,\delta\xi])}|\widehat{u_0}(\xi)|^2\langle\xi\rangle^{2s}d\xi. \tag{5.2}$$

Since $\partial^s\widehat{\varphi}(0) = 0$, the statement follows from the dominated convergence theorem.

Interpolating this inequality with the bound i), we obtain $\|u^\delta - u^{\delta'}\|_{H^s} = o(1)$ as δ, δ' go to zero. This implies that u^δ is a convergent sequence in $L^\infty([0,T];H^s)$. By (5.1),

$$\|u^\delta - u\|_{L^2} \lesssim \|u_0^\delta - u_0\|_{L^2} \to 0$$

as $\delta \to 0$, therefore u is the limit of u^δ also in H^s.

iii) Using the equation, we estimate

$$\partial_t \|\partial_x^s (u^\delta - v^\delta)\|_{L^2}^2 = 2\int \partial_x^s (u^\delta - v^\delta)_t \partial_x^s (u^\delta - v^\delta) dx$$
$$= -2\int \partial_x^{s+3}(u^\delta - v^\delta)\partial_x^s (u^\delta - v^\delta) dx - \int \partial_x^{s+1}((u^\delta)^2 - (v^\delta)^2)\partial_x^s (u^\delta - v^\delta) dx$$
$$\lesssim \|u^\delta - v^\delta\|_{H^s}^2 \big(\|u^\delta\|_{H^{s+1}} + \|v^\delta\|_{H^{s+1}}\big) \lesssim \frac{1}{\delta}\|u^\delta - v^\delta\|_{H^s}^2.$$

The first inequality follows since the first summand is zero and the second one can be estimated by considering the cases when $s+1$ derivatives hit $u^\delta + v^\delta$ and $u^\delta - v^\delta$. The second inequality follows from i).

This implies iii) by Gronwall's Lemma.

iv) Since $\|u_n^\delta - u_n\|_{H^s} = \lim_{\delta' \to 0} \|u_n^\delta - u_n^{\delta'}\|_{H^s}$, it suffices to prove that

$$\sup_n \|u_n^\delta - u_n^{\delta'}\|_{L^2} = o(\delta^s).$$

Interpolation with the bound i) yields the claim.

Using (5.1) and (5.2) it is enough to show that

$$\sup_n \int \|\partial^s \widehat{\varphi}\|_{L^\infty([0,\delta\xi])} |\widehat{u_{n,0}}(\xi)|^2 \langle \xi \rangle^{2s} d\xi = o(1).$$

Indeed,

$$\sup_n \int \|\partial^s \widehat{\varphi}\|_{L^\infty([0,\delta\xi])} |\widehat{u_{n,0}}|^2 \langle \xi \rangle^{2s} d\xi$$
$$\leq \int \|\partial^s \widehat{\varphi}\|_{L^\infty([0,\delta\xi])} |\widehat{u_0}|^2 \langle \xi \rangle^{2s} d\xi + \sup_n \int \|\partial^s \widehat{\varphi}\|_{L^\infty([0,\delta\xi])} |\widehat{u_{n,0}} - \widehat{u_0}|^2 \langle \xi \rangle^{2s} d\xi.$$

The first integral goes to zero by Lebesgue dominated convergence theorem. For the second, given $\epsilon > 0$, choose N so that $\|u_{n,0} - u_0\|_{H^s} < \epsilon$ for all $n > N$, and estimate the first N terms by dominated convergence theorem.

□

It remains to prove that the solutions constructed above can be defined globally-in-time. It is a well-known fact in the literature that smooth solutions of the KdV satisfy infinitely many conservation laws. A sample includes the following:

$$I_1(t) = \int u(x,t)dx = I_1(0)$$

$$I_2(t) = \int u^2(x,t)dx = I_2(0)$$

$$I_3(t) = \int \big(u_x^2 - \frac{1}{3}u^3(x)\big)dx = I_3(0)$$

which can be verified directly by taking the time derivative of the above quantities and show that $\partial_t I_j = 0$, $j = 1,2,3$. Each conservation law along with interpolation provides an a priori bound

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} \tag{5.3}$$

for s an integer. In general for a given H^s solution, to make sense of the time differentiation and then integration by parts, we need the solutions to live in a smoother than H^s space (H^{s+3} suffices for the KdV). We resolve this minor problem by considering smooth solutions as follows. Let $u_0 \in H^s$ and as before construct u_0^δ smooth such that $u_0^\delta \to u_0$ in H^s. By continuous dependence we know that $u^\delta \to u$ in H^s. Since u^δ is smooth it satisfies the a priori bound (5.3). But then

$$\|u\|_{H^s} \leq \|u^\delta - u\|_{H^s} + \|u^\delta\|_{H^s} \lesssim \|u^\delta - u\|_{H^s} + \|u_0^\delta\|_{H^s} \lesssim \|u_0\|_{H^s}$$

by taking $\delta \to 0$. Thus u satisfies the a priori bound. But then we can iterate the local solution and reach any time interval $[0,T]$. To see this assume that we have solve the problem locally-in-time and obtain a solution on $[0,T_1]$, $T_1 = f(\|u_0\|_{H^s})$, where on the same interval the solution satisfies (5.3). Now solve KdV with initial data $u(T_1)$ and obtain a solution on $[T_1,T_2]$ with $T_2 - T_1 = f(\|u(T_1)\|_{H^s}) = cf(\|u_0\|_{H^s})$ by the a priori bound (5.3). By continuity we can glue the solution together and thus u solves KdV on $[0,T_2]$ and on this interval now it satisfiies (5.3). Then $T_3 - T_2 = f(\|u(T_2)\|_{H^s}) = cf(\|u_0\|_{H^s})$. We continue with a uniform time step to cover $[0,T]$.

Remark 5.3. *Notice that we cannot iterate the a priori bound coming from the energy inequality. This is because, as the $\|u\|_{H^s}$ grows, going from one time interval to another, the time intervals shrinks. Thus it is possible that the sequence of times shrinks in such a way that it approaches a finite time limit and the process stops.*

5.2. **Kenig–Ponce–Vega method on $\mathbb{R}$.** We now describe a method that depends on the knowledge of the linear dispersive estimates. The theory was developed by Kenig-Ponce-Vega. Look at [47] and the references therein. The reader can also look at [55] for the application of the method to different PDE. Recall the definition of the Hilbert transform defined for functions in the Schwartz class:

$$Hf = \frac{1}{\pi}\text{p.v.}(\frac{1}{x} \star f) = \lim_{\epsilon\to 0}\int_{|y|\geq\epsilon} \frac{f(x-y)}{y}dy.$$

It is known that, [66], $\widehat{Hf}(\xi) = -isgn(\xi)\widehat{f}(\xi)$ and hence the operator is bounded on $L^2(\mathbb{R})$ with operator norm 1. Moreover it is known that

$$\|H(f)\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})} \text{ for } 1 < p < \infty.$$

We start with estimates for the linear KdV

$$u_t + u_{xxx} = 0, \quad u(0,x) = u_0(x).$$

Taking space Fourier transform we obtain

$$\widehat{u}(\xi,t) = e^{it\xi^3}\widehat{u_0}(\xi)$$

and thus

$$u(x,t) = \int_{\mathbb{R}} e^{i(x\xi + t\xi^3)}\widehat{u_0}(\xi)d\xi.$$

The solution $W(t)u_0 = A_t \star u_0(x)$ is given by convolution with the Airy kernel

$$A_t(x) = \int e^{it\xi^3 + ix\xi}d\xi.$$

Note that $\|W(t)u_0\|_{H^s} = \|u_0\|_{H^s}$ for all real s. We have the following dispersive decay estimate:

Lemma 5.4. *For* $\alpha \in [0, 1/2]$*,*

$$D^\alpha A_t(x) = \int e^{it\xi^3 + ix\xi} |\xi|^\alpha d\xi$$

satisfies the bound

$$\|D^\alpha A_t\|_{L^\infty} \lesssim |t|^{-(\alpha+1)/3}.$$

Proof. By the scaling relation

$$|D^\alpha A_t(x)| = |t|^{-(\alpha+1)/3} |D^\alpha A_t(x/t^{1/3})|$$

it suffices to prove that $|D^\alpha A_1(x)|$ is a bounded function. Since the estimate is trivial on the interval $|\xi| < 2$, it suffices to consider the integral

$$\int e^{i\xi^3 + ix\xi} \psi(\xi) |\xi|^\alpha d\xi,$$

where ψ is a C^∞ function satisfying $\psi(\xi) = 0$ for $|\xi| < 1$ and $\psi(\xi) = 1$ for $|\xi| \geq 2$. If $x > -1$, the estimate follows from one integration by parts by writing

$$e^{i\xi^3 + i\xi x} = \frac{1}{3i\xi^2 + ix} \partial_\xi e^{i\xi^3 + i\xi x}$$

since then

$$\frac{|\xi|^\alpha}{|3\xi^2 + x|} \lesssim 1.$$

For $x < -1$ divide the integral into two pieces using smooth cutoffs ψ_1 and ψ_2 ($\psi_1 + \psi_2$=1) where ψ_1 is supported on the set

$$A = \{\xi : |3\xi^2 + x| < |x|/2\}$$

and ψ_2 is supported on

$$B = \{\xi : |3\xi^2 + x| > |x|/3\}.$$

To estimate the contribution of ψ_2 integrate by parts to obtain the bound

$$\int \Big| \frac{d}{d\xi} \frac{|\xi|^\alpha \psi(\xi) \psi_2(\xi)}{3\xi^2 + x} \Big| d\xi.$$

Since on the set B, $|3\xi^2 - x| \leq |3\xi^2 + x| + 2|x| \lesssim |3\xi^2 + x|$, we obtain $|3\xi^2 + x| \gtrsim 3\xi^2 + |x| \gtrsim \langle \xi \rangle^2$. Also note that the derivative inside the integral can change sign at most a finite number of times. Therefore by fundamental theorem of calculus it suffices to see that

$$\Big| \frac{|\xi|^\alpha \psi(\xi) \psi_2(\xi)}{3\xi^2 + x} \Big| \lesssim \frac{|\xi|^\alpha}{\langle \xi \rangle^2} \lesssim 1.$$

To estimate the contribution of ψ_1, note that when $\xi \in A$, $\xi^2 \approx |x|$, and hence the second derivative of the phase, $\xi^3 + \xi x$, is $\gtrsim |x|^{1/2}$. Therefore by Van der Corput Lemma we estimate the contribution of ψ_1 by

$$|x|^{-1/4} \Big(\big\| |\xi|^\alpha \psi_1(\xi) \psi(\xi) \big\|_{L^\infty} + \big\| \frac{d}{d\xi} \big(|\xi|^\alpha \psi_1(\xi) \psi(\xi) \big) \big\|_{L^1} \Big).$$

Since the derivative changes sign at most finitely many times the two norms have the same contribution $\lesssim |x|^{\alpha/2}$. Therefore for $\alpha \in [0, 1/2]$, we obtain a uniform bound. $\square$

Theorem 5.5. *(Dispersive decay estimate) For any $\theta \in [0,1]$ and $\alpha \in [0,1/2]$, we have*

$$\|D^{\alpha\theta} W_t u_0\|_{L^{2/(1-\theta)}} \lesssim |t|^{-\theta(\alpha+1)/3} \|u_0\|_{L^{2/(1+\theta)}}$$

Proof. Writing

$$W(t)u_0 = A_t * u_0 = \int e^{it\xi^3 + i(x-y)\xi} u_0(y) d\xi dy,$$

the lemma above implies that

$$\|D^{\alpha} W_t u_0\|_{L^\infty} \lesssim |t|^{-(\alpha+1)/3} \|u_0\|_{L^1}.$$

The theorem will follow from complex interpolation[4] of this bound with the L^2 conservation bound. To do this consider the analytic family of operators

$$D^z W(t) u_0 = D^z A_t * u_0$$

where $z = \alpha + i\beta$, $\alpha \in [0,1/2]$, $\beta \in \mathbb{R}$. Since $D^{i\beta}$ is unitary, the operator is uniformly bounded in L^2 for $\alpha = 0$. Repeating the proof of the lemma above with $|\xi|^{\alpha+i\beta}$ instead of $|\xi|^\alpha$ gives

$$\|D^{\alpha+i\beta} W_t u_0\|_{L^\infty} \lesssim \langle\beta\rangle |t|^{-(\alpha+1)/3} \|u_0\|_{L^1}.$$

Therefore complex interpolation between the lines $\Re(z) = 0$ and $\Re(z) = \alpha$ yield the theorem. $\square$

Theorem 5.6. *(Strichartz estimates)* [69], [49]. *For any $\theta \in [0,1]$ and $\alpha \in [0,1/2]$, we have*

$$\|D^{\alpha\theta/2} W_t u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2} \tag{5.4}$$

$$\Big\| \int_0^t D^{\alpha\theta} W_{t-\tau} g(\cdot,\tau) d\tau \Big\|_{L^q_t L^r_x} \lesssim \|g\|_{L^{q'}_t L^{r'}_x}, \tag{5.5}$$

where $(q,r) = \big(6/(\theta(\alpha+1)), 2/(1-\theta)\big)$.

Proof. As usual by the TT^* argument (with $T = D^{\alpha\theta/2} W_t$) (5.4) follows from the bound

$$\begin{aligned}\Big\| \int_{\mathbb{R}} D^{\alpha\theta} W_{t-\tau} g(\cdot,\tau) d\tau \Big\|_{L^q_t L^r_x} &\le \Big\| \int_{\mathbb{R}} \|D^{\alpha\theta} W_{t-\tau} g(\cdot,\tau)\|_{L^r_x} d\tau \Big\|_{L^q_t} \\ &\lesssim \Big\| \int_{\mathbb{R}} |t-\tau|^{-\theta(\alpha+1)/3} \|g\|_{L^{r'}_x} d\tau \Big\|_{L^q_t} \lesssim \|g\|_{L^{q'}_t L^{r'}_x}.\end{aligned}$$

Here, we used Minkowski integral inequality, the dispersive bound above and fractional integration in that order. The inequality (5.5) is proved similarly:

$$\begin{aligned}&\Big\| \int_0^t D^{\alpha\theta} W_{t-\tau} g(\cdot,\tau) d\tau \Big\|_{L^q_t L^r_x} \le \Big\| \int_0^t \|D^{\alpha\theta} W_{t-\tau} g(\cdot,\tau)\|_{L^r_x} d\tau \Big\|_{L^q_t} \\ &\lesssim \Big\| \int_0^t |t-\tau|^{-\theta(\alpha+1)/3} \|g\|_{L^{r'}_x} d\tau \Big\|_{L^q_t} \le \Big\| \int_{\mathbb{R}} |t-\tau|^{-\theta(\alpha+1)/3} \|g\|_{L^{r'}_x} d\tau \Big\|_{L^q_t} \lesssim \|g\|_{L^{q'}_t L^{r'}_x}.\end{aligned}$$

$\square$

[4] This theorem is due to Elias Stein and is an extension of the Riesz-Thorin theorem. In this case the linear operators depend analytically on a parameter z, see [67].

Note that in particular we have the bounds

$$\|D^{1/4}W_t u_0\|_{L^4_t L^\infty_x} \lesssim \|u_0\|_{L^2}. \tag{5.6}$$

$$\|W_t u_0\|_{L^8_t L^8_x} \lesssim \|u_0\|_{L^2}. \tag{5.7}$$

Theorem 5.7. *(Kato Smoothing)*

$$\|\partial_x W_t u_0\|_{L^\infty_x L^2_t} \lesssim \|u_0\|_{L^2}.$$

Proof. Writing

$$\partial_x W_t u_0 = i\int \xi e^{i\xi^3 t + i\xi x}\widehat{u_0}(\xi)d\xi \overset{\eta=\xi^3}{=} \frac{i}{3}\int \eta^{-1/3}e^{i\eta t + i\eta^{1/3}x}\widehat{u_0}(\eta^{1/3})d\eta,$$

we see that (by Plancherel's Theorem)

$$\|\partial_x W_t u_0\|_{L^2_t} = \frac{1}{3}\|\eta^{-1/3}e^{i\eta^{1/3}x}\widehat{u_0}(\eta^{1/3})\|_{L^2_\eta} \overset{\xi=\eta^{1/3}}{=} \|\widehat{u_0}\|_{L^2} = \|u_0\|_{L^2}.$$

□

Finally we state without proof a maximal function inequality:

Theorem 5.8. *For any $s > 3/4$,*

$$\|W_t u_0\|_{L^2_x L^\infty_{t\in[-T,T]}} \lesssim \langle T\rangle^s \|u_0\|_{H^s}.$$

We now establish the local wellposedness of the KdV equation $u_t + u_{xxx} + uu_x = 0$ on the real line for $\frac{3}{4} < s < 1$ (the same method works for all $s > \frac{3}{4}$). The method is not suitable for nonlinear dispersive PDEs on bounded domains since it is based on the use of the basic dispersive estimates we outlined above. First we need a Lemma whose proof we skip but the reader can consult [55] and the references therein for similar commutator estimates:

Lemma 5.9. *For $s \in (0,1)$, we have*

$$\|J^s(fg) - fJ^s g\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{L^\infty}.$$

For $s > 1$, we have

$$\left\|J^s(fg) - fJ^s g\right\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{L^\infty} + \|f_x\|_{L^\infty}\|g\|_{H^{s-1}}$$

where the operators J^s given on the Fourier side as

$$\widehat{J^s f}(\xi) = \langle\xi\rangle^s \widehat{f}(\xi).$$

To motivate the choice for the space X we work with, we start with the following:

Lemma 5.10. *For $s \in (\frac{1}{2}, 1)$ and $0 < \delta \lesssim 1$, we have*

$$\|uu_x\|_{L^2_{t\in[0,\delta]}H^s_x} \lesssim \delta^{1/4}\|u_x\|_{L^4_t L^\infty_x}\|u\|_{L^\infty_t H^s_x} + \|u\|_{L^2_x L^\infty_t}\|D^s u_x\|_{L^\infty_x L^2_t} + \delta^{\frac{1}{2}}\|u\|^2_{L^\infty_t H^s_x}.$$

Proof. First note that

$$\|uu_x\|_{H^s_x} = \|J^s(uu_x)\|_{L^2_x} \leq \|J^s(uu_x) - uJ^s u_x\|_{L^2_x} + \|uJ^s u_x\|_{L^2_x} =: I + II.$$

Using Lemma 5.9, we obtain

$$I \lesssim \|u_x\|_{L^\infty_x}\|u\|_{H^s}.$$

Note that

$$II \leq \|uD^s u_x\|_{L^2_x} + \|u(J^s - D^s)u_x\|_{L^2_x}$$
$$\lesssim \|uD^s u_x\|_{L^2_x} + \|u\|_{L^\infty_x} \|u\|_{L^2_x} \lesssim \|uD^s u_x\|_{L^2_x} + \|u\|^2_{H^s}.$$

The second inequality follows from the boundedness of the multiplier (recall that $s \in [0,1]$)

$$m(\xi) = \xi[(1+|\xi|^2)^{\frac{s}{2}} - |\xi|^s],$$

and the third by Sobolev embedding. Combining these bounds and then using Hölder's inequality yield the statement

$$\|uu_x\|_{L^2_{t\in[0,\delta]}H^s_x} \lesssim \big\|\|u_x\|_{L^\infty_x}\|u\|_{H^s}\big\|_{L^2_{t\in[0,\delta]}} + \big\|\|u\|^2_{H^s}\big\|_{L^2_{t\in[0,\delta]}} + \|uD^s u_x\|_{L^2_x L^2_{t\in[0,\delta]}}$$
$$\lesssim \delta^{1/4}\|u_x\|_{L^4_t L^\infty_x}\|u\|_{L^\infty_t H^s_x} + \delta^{\frac{1}{2}}\|u\|^2_{L^\infty_t H^s_x} + \|u\|_{L^2_x L^\infty_t}\|D^s u_x\|_{L^\infty_x L^2_t}.$$

Note that we changed the order of the norms in the last term of the first line. □

Now we are ready to prove that the KdV equation is locally wellposed on $H^s(\mathbb{R})$ for $s > 3/4$. To do that, we apply Banach's fixed point theorem on the ball (for sufficiently small $\delta = \delta(\|g\|_{H^s})$)

$$B_\delta = \{u \in X \cap C^0_t H^s_x([0,\delta]\times\mathbb{R}) : \|u\|_X \leq M\|g\|_{H^s}\},$$

to the operator

$$\Gamma u(t) = W_t g(x) - \int_0^t W_{t-\tau}\big(u(\tau)u_x(\tau)\big)d\tau,$$

where

$$\|u\|_X = \max(\|u\|_{L^\infty_{t\in[0,\delta]}H^s_x}, \|u_x\|_{L^4_{t\in[0,\delta]}L^\infty_x}, \|u\|_{L^2_x L^\infty_{t\in[0,\delta]}}, \|D^s u_x\|_{L^\infty_x L^2_{t\in[0,\delta]}}).$$

Recall that $W_t = e^{-t\partial_x^3}$, W_t is unitary in H^s spaces, and that W_t commutes with J^s, D^s, and the Hilbert transform H. Note that the norms appearing in the definition of X are the ones on the right-hand side of the inequality in Lemma 5.10. We only prove that Γ maps B_δ into itself provided that $\delta = \delta(\|g\|_{H^s})$ is sufficiently small and M is sufficiently large. Estimates on the differences can be obtained similarly. We start with $\|\Gamma u\|_{L^\infty_{t\in[0,\delta]}H^s_x}$

$$\|\Gamma u\|_{L^\infty_t H^s_x} \lesssim \|g\|_{H^s} + \|\int_0^t W_{t-\tau}(uu_x)d\tau\|_{L^\infty_t H^s_x}$$
$$\leq \|g\|_{H^s} + \|\int_0^t \|uu_x\|_{H^s_x}d\tau\|_{L^\infty_t}$$
$$\lesssim \|g\|_{H^s} + \delta^{1/2}\|uu_x\|_{L^2_t H^s_x} \lesssim \|g\|_{H^s} + \delta^{1/2}\|u\|^2_X.$$

We used Lemma 5.10 in the last inequality. Similarly, using $\partial_x = DH$, where H is the Hilbert transform, and the fact that W_t commutes with D and H, we obtain

$$\begin{aligned}
\|\partial_x \Gamma u\|_{L^4_{t\in[0,\delta]}L^\infty_x} &\lesssim \|\partial_x W_t g\|_{L^4_t L^\infty_x} + \|\partial_x \int_0^t W_{t-\tau}(uu_x)d\tau\|_{L^4_t L^\infty_x} \\
&\leq \|D^{1/4} W_t D^{3/4} H g\|_{L^4_t L^\infty_x} + \| \int_0^t \|D^{1/4} W_t W_{-\tau} D^{3/4} H(uu_x)\|_{L^\infty_x} d\tau \|_{L^4_t} \\
&\leq \|D^{3/4} H g\|_{L^2_x} + \int_0^\delta \|D^{1/4} W_t W_{-\tau} D^{3/4} H(uu_x)\|_{L^4_t L^\infty_x} d\tau \\
&\leq \|D^{3/4} H g\|_{L^2_x} + \int_0^\delta \|D^{3/4} H(uu_x)\|_{L^2_x} d\tau \\
&\lesssim \|g\|_{H^s} + \delta^{1/2} \|uu_x\|_{L^2_t H^s_x} \lesssim \|g\|_{H^s} + \delta^{1/2} \|u\|_X^2.
\end{aligned}$$

In the third and forth inequalities, we used Strichartz inequality (5.6), and in the last inequality, we used Lemma 5.10. By the maximal function inequality (Theorem 5.8), we have

$$\begin{aligned}
\|\Gamma u\|_{L^2_x L^\infty_{t\in[0,\delta]}} &\leq \|W_t g\|_{L^2_x L^\infty_t} + \| \int_0^t |W_{t-\tau}(uu_x)| d\tau \|_{L^2_x L^\infty_t} \\
&\lesssim \|g\|_{H^s} + \int_0^\delta \|W_t W_{-\tau}(uu_x)\|_{L^2_x L^\infty_t} d\tau \\
&\lesssim \|g\|_{H^s} + \int_0^\delta \|uu_x\|_{H^s_x} d\tau \\
&\lesssim \|g\|_{H^s} + \delta^{1/2} \|uu_x\|_{L^2_t H^s_x} \lesssim \|g\|_{H^s} + \delta^{1/2} \|u\|_X^2.
\end{aligned}$$

Finally, we estimate $\|D^s \partial_x \Gamma u\|_{L^\infty_x L^2_{t\in[0,\delta]}}$ using the Kato smoothing estimate (Theorem 5.7)

$$\begin{aligned}
\|D^s \partial_x \Gamma u\|_{L^\infty_x L^2_{t\in[0,\delta]}} &\leq \|\partial_x W_t D^s g\|_{L^\infty_x L^2_t} + \| \int_0^t |D^s \partial_x W_{t-\tau}(uu_x)| d\tau \|_{L^\infty_x L^2_t} \\
&\lesssim \|g\|_{H^s} + \int_0^\delta \|\partial_x W_t W_{-\tau} D^s(uu_x)\|_{L^\infty_x L^2_t} d\tau \\
&\lesssim \|g\|_{H^s} + \int_0^\delta \|uu_x\|_{H^s_x} d\tau \\
&\lesssim \|g\|_{H^s} + \delta^{1/2} \|uu_x\|_{L^2_t H^s_x} \lesssim \|g\|_{H^s} + \delta^{1/2} \|u\|_X^2.
\end{aligned}$$

We thus obtain the following estimate

$$\|\Gamma u\|_X \lesssim \|g\|_{H^s} + \delta^{1/2} \|u\|_X^2.$$

Therefore, one can close the argument by choosing M large and δ small, provided that $\Gamma u \in C^0_t H^s_x([0,\delta] \times \mathbb{R})$. We only prove the continuity at time 0. The proof is similar for each time because of the group structure of W_t. By the bounds above and the continuity of W_t in H^s, we have

$$\|\Gamma u(t) - g\|_{H^s} \lesssim \|W_t g - g\|_{H^s} + t^{1/2} \|u\|_X^2 \to 0, \quad \text{as } t \to 0,$$

which finishes the proof. Since we have proved the local wellposedness by a contraction argument, the method also implies continuous (Lipschitz) dependence on the initial data.

Remark 5.11. *Suppose that one proves existence and uniqueness in B_δ which is a fixed ball in the space X. One can then easily extend the uniqueness to the whole space X by shrinking time by a fixed amount. Indeed, by shrinking time to δ' we get existence and uniqueness in a larger ball $B_{\delta'}$. Now assume that there are two different solutions, one staying in the ball B_δ and one separating after hitting the boundary at some time $|t| < \delta'$. This is already a contradiction by the uniqueness in $B_{\delta'}$.*

Finally, the H^1 a priori bound coming from the conservation laws extends this local solution globally-in-time in H^1 as described in the previous section. We note that the oscillatory integral method is very efficient on unbounded domains where the dispersion is in full effect, and it has been applied to various dispersive models with derivative nonlinearities, such as the generalized KdV equations, derivative NLS equations, and Zakharov system; see, e.g., Kenig–Ponce–Vega [47, 50, 51]. As mentioned above, one drawback of this method is the fact that it relies on the dispersive estimates that are not true over compact domains, in particular over $\mathbb{T}$. In the next section, we demonstrate a method that can be used to study local well–posedness on both $\mathbb{R}$ and $\mathbb{T}$.

5.3. **Restricted norm method of Bourgain.**

5.3.1. *KdV on* $\mathbb{R}$. In this section we outline a method that was developed by Bourgain in [9, 7, 8]. For further applications of the method see also [48]. Let $X^{s,b}$ be the Banach space of functions on $\mathbb{R}\times\mathbb{R}$ (or $\mathbb{T}\times\mathbb{R}$) defined by the norm

$$\|u\|_{X^{s,b}} = \left\|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\widehat{u}(\xi,\tau)\right\|_{L^2_{\xi,\tau}} = \left\|W_{-t}u\right\|_{H^s_xH^b_t}.$$

Since the contraction argument will be in a time interval $[-\delta,\delta]$ with $\delta\le 1$, we also define the restricted $X^{s,b}$ space, $X^{s,b}_\delta$, as the equivalent classes of functions that agree on $[-\delta,\delta]$ with the norm

$$\|u\|_{X^{s,b}_\delta} = \inf_{\tilde{u}=u,\, t\in[-\delta,\delta]}\|\tilde{u}\|_{X^{s,b}}.$$

The contraction will be for the operator

$$\Phi u = \eta(t)W_tu_0 - \eta(t)\int_0^t W_{t-s}(uu_x)ds,$$

where η is a C_0^∞ function satisfying $\eta(t)=1, t\in[-1,1]$. Since $\delta\le 1$, a fix point of Φ gives us a solution of KdV on $[-\delta,\delta]$. We will only discuss the case $s=0$, which implies that the solution is globally defined for all times due to L^2 conservation. Similar ideas can push the local well-posedness to any $s\ge -3/4$.

Lemma 5.12. *For $\delta\le 1$, $s,b\in\mathbb{R}$, we have*

$$\|\eta(t)W_tu_0\|_{X^{s,b}_\delta}\lesssim\|u_0\|_{H^s}.$$

Proof.

$\|\eta(t)W_t u_0\|_{X^{s,b}_\delta} \leq \|W_t\eta(t)u_0\|_{X^{s,b}} = \|W_{-t}W_t\eta(t)u_0\|_{H^s_x H^b_t} = \|\eta\|_{H^b}\|u_0\|_{H^s} \lesssim \|u_0\|_{H^s}.$

The first inequality follows from the definition of the restricted norm and the fact that W_t and $\eta(t)$ commute. □

Lemma 5.13. *For any $b > 1/2$, $X^{s,b}$ embedds into $C(\mathbb{R}; H^s)$.*

Proof. We will do proof for the real line, the proof is the same on the torus. By Fourier inversion and then a change of variable we have

$$\begin{aligned} u(t,x) &= \int\int \widehat{u}(\xi,\tau)e^{it\tau+ix\xi}d\xi d\tau \\ &= \int\int e^{it\xi^3}\widehat{u}(\xi,\tau+\xi^3)e^{it\tau+ix\xi}d\xi d\tau \\ &= \int e^{it\tau}W_t\psi_\tau d\tau, \end{aligned}$$

where $\widehat{\psi_\tau}(\xi) = \widehat{u}(\xi,\tau+\xi^3)$. Therefore for each t

$$\begin{aligned} \|u\|_{H^s_x} &\leq \int \|W_t\psi_\tau\|_{H^s}d\tau = \int \|\psi_\tau\|_{H^s}d\tau \\ &= \int \left\|\widehat{u}(\xi,\tau+\xi^3)\langle\xi\rangle^s\right\|_{L^2_\xi}d\tau \\ &\leq \|\langle\tau\rangle^{-b}\|_{L^2_\tau}\left\|\widehat{u}(\xi,\tau+\xi^3)\langle\xi\rangle^s\langle\tau\rangle^b\right\|_{L^2_\xi L^2_\tau} \\ &\lesssim \|u\|_{X^{s,b}}. \end{aligned}$$

Continuity in t follows from this, continuity of the linear group and the dominated convergence theorem. □

Lemma 5.14. *For any $-1/2 < b' < b < 1/2$ and $s \in \mathbb{R}$, we have*

$$\|u\|_{X^{s,b'}_\delta} \lesssim \delta^{b-b'}\|u\|_{X^{s,b}_\delta}.$$

Proof. We will give the proof for $0 \leq b' < b < 1/2$. By duality this implies the inequality for $-1/2 < b' < b \leq 0$ as follows

$$\|u\|_{X^{s,b'}_\delta} = \sup_{\|g\|_{X^{-s,-b'}_\delta}=1}\left|\int ug\right| \leq \sup_{\|g\|_{X^{-s,-b'}_\delta}=1}\|u\|_{X^{s,b}_\delta}\|g\|_{X^{-s,-b}_\delta} \lesssim \delta^{b-b'}\|u\|_{X^{s,b}_\delta}.$$

By combining these two inequalities we get the full range. To obtain the inequality for $0 \leq b' < b < 1/2$, first note that by replacing u with $J^s u$ we can assume that $s = 0$. Second, by definition of the restricted norm it suffices to prove that (by taking infimum over $\tilde{u}$)

$$\|\eta(t/\delta)\tilde{u}\|_{X^{0,b'}} \lesssim \delta^{b-b'}\|\tilde{u}\|_{X^{0,b}}.$$

Suppresing the $\tilde{u}$ notation, we have

$$\|\eta(t/\delta)u\|_{X^{0,b'}} = \|\eta(t/\delta)W_{-t}u\|_{L^2_x H^{b'}_t}.$$

Therefore it suffices to prove that

$$\|\eta(t/\delta)f(t)\|_{H^{b'}} \lesssim \delta^{b-b'}\|f\|_{H^b}.$$

Using $\langle\tau\rangle^{b'} \leq \langle\tau-\tau_1\rangle^{b'} + \langle\tau_1\rangle^{b'}$, we obtain (with $\frac{1}{p_1} = b-b'$, $\frac{1}{p_2} = b$, $\frac{1}{q_1} = \frac{1}{2}+b'-b$, $\frac{1}{q_2} = \frac{1}{2}-b$)

$$\begin{aligned}
\|\eta(t/\delta)f(t)\|_{H^{b'}} &\leq \|\eta(t/\delta)J^{b'}f\|_{L^2} + \|fJ^{b'}\eta(t/\delta)\|_{L^2}\\
&\leq \|\eta(t/\delta)\|_{L^{p_1}}\|J^{b'}f\|_{L^{q_1}} + \|f\|_{L^{q_2}}\|J^{b'}\eta(t/\delta)\|_{L^{p_2}}\\
&\lesssim \|f\|_{H^b}\big(\|\eta(t/\delta)\|_{L^{p_1}} + \|J^{b'}\eta(t/\delta)\|_{L^{p_2}}\big)\\
&\lesssim \|f\|_{H^b}\big(\delta^{b-b'} + \|\eta(t/\delta)\|_{H^{\frac{1}{2}-b+b'}}\big)\\
&\lesssim \delta^{b-b'}\|f\|_{H^b}.
\end{aligned}$$

In the last two inequalities we used Sobolev embedding. □

Lemma 5.15. [30] *Let* $-\frac{1}{2} < b' \leq 0$ *and* $b = b'+1$. *Then*

$$\Big\|\eta(t)\int_0^t W_{t-s}F(s)ds\Big\|_{X^{s,b}_\delta} \lesssim \|F\|_{X^{s,b'}_\delta}.$$

Proof. As before it suffices to prove the statement with $X^{s,b}$ norms. Note that

$$\Big\|\eta(t)\int_0^t W_{t-s}F(s)ds\Big\|_{X^{s,b}} = \Big\|\eta(t)\int_0^t W_{-s}F(s)ds\Big\|_{H^s_xH^b_t}.$$

Therefore it suffices to prove that

$$\Big\|\eta(t)\int_0^t f(s)ds\Big\|_{H^b} \lesssim \|f\|_{H^{b'}}. \tag{5.8}$$

Writing

$$\int_0^t f(s)ds = \int \chi_{[0,t]}(s)f(s)ds = \int \chi^{\vee}_{[0,t]}(z)\widehat{f}(z)dz = \int \frac{e^{izt}-1}{iz}\widehat{f}(z)dz,$$

we see that the Fourier transform of the function inside the norm of the left hand side of (5.8) is

$$\int \frac{\widehat{\eta}(\tau-z)-\widehat{\eta}(\tau)}{iz}\widehat{f}(z)dz = \int_{|z|<1} + \int_{|z|>1}.$$

For the contribution of the first integral to the left hand side of (5.8) we use the mean value theorem to get

$$\begin{aligned}
\Big\|\langle\tau\rangle^b\int_{|z|<1}\Big\|_{L^2} &\lesssim \Big\|\int_{|z|<1}\langle\tau\rangle^b\sup_{|\tau'-\tau|<1}|\widehat{\eta}'(\tau')|\,|\widehat{f}(z)|dz\Big\|_{L^2}\\
&= \big\|\langle\tau\rangle^b\sup_{|\tau'-\tau|<1}|\widehat{\eta}'(\tau')|\big\|_{L^2}\int_{|z|<1}|\widehat{f}(z)|dz\\
&\lesssim \sqrt{\int_{|z|<1}|\widehat{f}(z)|^2dz} \lesssim \|f\|_{H^{b'}}.
\end{aligned}$$

For the contribution of the second integral we use the inequality $\langle\tau\rangle^b \lesssim \langle\tau - z\rangle^b\langle z\rangle^b$ and Young's inequality to get

$$\begin{aligned}
\Big\|\langle\tau\rangle^b\int_{|z|>1}\Big\|_{L^2} &\le \Big\|\langle\tau\rangle^b\int\frac{|\widehat{\eta}(\tau-z)|+|\widehat{\eta}(\tau)|}{\langle z\rangle}|\widehat{f}(z)|dz\Big\|_{L^2}\\
&\lesssim \Big\|\int\Big(\frac{\langle\tau-z\rangle^b|\widehat{\eta}(\tau-z)|}{\langle z\rangle^{1-b}}+\frac{\langle\tau\rangle^b|\widehat{\eta}(\tau)|}{\langle z\rangle}\Big)|\widehat{f}(z)|dz\Big\|_{L^2}\\
&\lesssim \big\|\langle\tau\rangle^b\widehat{\eta}\big\|_{L^1}\big\|\langle z\rangle^{b-1}\widehat{f}\big\|_{L^2}+\big\|\langle\tau\rangle^b\widehat{\eta}\big\|_{L^2}\big\|\langle z\rangle^{-1}\widehat{f}\big\|_{L^1}\\
&\lesssim \big\|\langle z\rangle^{b'}\widehat{f}\big\|_{L^2}+\big\|\langle z\rangle^{b'}\widehat{f}\big\|_{L^2}\big\|\langle z\rangle^{-1-b'}\big\|_{L^2}\lesssim\|f\|_{H^{b'}}.
\end{aligned}$$

The last inequality follows from the fact that $-1-b' < -1/2$. □

Remark 5.16. *Note that for $b = 1/2$, the proof above implies that*

$$\Big\|\eta(t)\int_0^t f(s)ds\Big\|_{H^{1/2}} \lesssim \|f\|_{H^{-1/2}} + \|\langle z\rangle^{-1}\widehat{f}\|_{L^1}. \tag{5.9}$$

Lemma 5.17. *For any $b > \frac{1}{2}$ and $b_1 \ge 1/4$ we have*

$$\|\partial_x u^2\|_{X_\delta^{0,-b_1}} \lesssim \|u\|^2_{X_\delta^{0,b}}.$$

Proof. Once again we can ignore δ dependence and work with $X^{s,b}$ norms. By duality, it suffices to prove that

$$\Big|\int g\partial_x u^2 dxdt\Big| \lesssim \|u\|^2_{X^{0,b}}\|g\|_{X^{0,b_1}}.$$

Using the Fourier multiplication formula,

$$\int fg = \int \widehat{f}g^\vee,$$

and renaming the variables, we write the left hand side as

$$\Big|\int_{\mathbb{R}^2}\xi\widehat{u^2}(\xi,\tau)\widehat{g}(-\xi,-\tau)d\tau d\xi\Big| = \Big|\int_{\substack{\xi_1+\xi_2+\xi_3=0\\ \tau_1+\tau_2+\tau_3=0}}\xi_3\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi_2,\tau_2)\widehat{g}(\xi_3,\tau_3)\Big|$$

Using the notation

$$\begin{aligned}
f_1(\xi,\tau) = f_2(\xi,\tau) &= |\widehat{u}(\xi,\tau)|\langle\tau-\xi^3\rangle^b,\\
f_3(\xi,\tau) &= |\widehat{g}(-\xi,-\tau)|\langle\tau-\xi^3\rangle^{b_1},
\end{aligned}$$

it suffices to prove that

$$\int\frac{|\xi|f_1(\xi_1,\tau_1)f_2(\xi-\xi_1,\tau-\tau_1)f_3(\xi,\tau)}{\langle\tau_1-\xi_1^3\rangle^b\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^b\langle\tau-\xi^3\rangle^{b_1}}d\xi d\xi_1 d\tau d\tau_1 \lesssim \prod_{i=1}^3\|f_i\|_2. \tag{5.10}$$

We claim that

$$\sup_{\xi,\tau}\frac{|\xi|^2}{\langle\tau-\xi^3\rangle^{2b_1}}\int\frac{1}{\langle\tau_1-\xi_1^3\rangle^{2b}\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}}d\xi_1 d\tau_1 \lesssim 1. \tag{5.11}$$

By using the Cauchy-Scwarz inequality in ξ_1, τ_1 integrals and using the claim we estimate (5.10) by

$$\int \Big(\int f_1^2(\xi_1,\tau_1) f_2^2(\xi-\xi_1,\tau-\tau_1) d\xi_1 d\tau_1 \Big)^{1/2} f_3(\xi,\tau) d\xi d\tau$$

$$\leq \Big(\int f_1^2(\xi_1,\tau_1) f_2^2(\xi-\xi_1,\tau-\tau_1) d\xi_1 d\tau_1 d\xi d\tau \Big)^{1/2} \Big(\int f_3^2(\xi,\tau) d\xi d\tau \Big)^{1/2} = \prod_{i=1}^{3} \|f_i\|_2.$$

It remains to prove (5.11). Using the estimate (for $b > 1/2$)

$$\int_{\mathbb{R}} \frac{1}{\langle x-\alpha\rangle^{2b} \langle x-\beta\rangle^{2b}} dx \lesssim \frac{1}{\langle \alpha-\beta\rangle^{2b}}$$

in the τ_1 integral we bound (5.11) by

$$\sup_{\xi,\tau} \frac{|\xi|^2}{\langle \tau-\xi^3\rangle^{2b_1}} \int \frac{1}{\langle \tau-\xi_1^3-(\xi-\xi_1)^3\rangle^{2b}} d\xi_1. \tag{5.12}$$

Let $x = \tau - \xi_1^3 - (\xi-\xi_1)^3$. Using

$$\xi_1 = \frac{3\xi^2 \pm \sqrt{3\xi(4\tau-\xi^3-4x)}}{6\xi},$$

we obtain

$$dx = (3\xi^2 - 6\xi\xi_1) d\xi = \pm\sqrt{3\xi(4\tau-\xi^3-4x)} d\xi_1.$$

Therefore, we can estimate (5.12) by

$$\sup_{\xi,\tau} \frac{|\xi|^2}{\langle \tau-\xi^3\rangle^{2b_1}} \int \frac{1}{\langle x\rangle^{2b} \sqrt{|\xi|} \sqrt{|4\tau-\xi^3-4x|}} dx.$$

Using the inequality (for $b > 1/2$)

$$\int_{\mathbb{R}} \frac{1}{\langle x\rangle^{2b} \sqrt{|x-\beta|}} dx \lesssim \frac{1}{\langle \beta\rangle^{1/2}},$$

we obtain (for $b_1 \geq 1/4$)

$$\sup_{\xi,\tau} \frac{|\xi|^{3/2}}{\langle \tau-\xi^3\rangle^{2b_1} \langle 4\tau-\xi^3\rangle^{1/2}} \lesssim 1.$$

□

We now run the contraction argument in $X_\delta^{0,b}$ (with $b > 1/2$ and δ sufficiently small) for the operator

$$\Phi u = \eta(t) W_t u_0 - \eta(t) \int_0^t W_{t-s}(uu_x) ds.$$

Using the bounds in Lemma 5.12 and in Lemma 5.15, we have

$$\|\Phi u\|_{X_\delta^{0,b}} \lesssim \|u_0\|_{L^2} + \|uu_x\|_{X_\delta^{0,b-1}}.$$

Now, using Lemma 5.14 (with $b_1 \geq 1/4$) and then Lemma 5.17, we obtain

$$\begin{aligned} \|\Phi u\|_{X_\delta^{0,b}} &\lesssim \|u_0\|_{L^2} + \delta^{1-b-b_1} \|uu_x\|_{X_\delta^{0,-b_1}} \\ &\lesssim \|u_0\|_{L^2} + \delta^{1-b-b_1} \|u\|^2_{X_\delta^{0,b}}. \end{aligned}$$

Therefore we can close the contraction for any $b > 1/2$, $b_1 \geq 1/4$, $1-b-b_1 > 0$ by choosing $\delta = \delta(\|u_0\|_{L^2}, b, b_1)$ sufficiently small.

5.3.2. *KdV on* $\mathbb{T}$. In what follows we will consider mean zero solutions of the KdV equation. This assumption can be justified as follows (we note that on $\mathbb{R}$ this idea fails): Let $u_t + u_{xxx} + uu_x = 0$ with $u(x,0) = u_0$. If we integrate the equation in space we obtain $\partial_t \int_{\mathbb{T}} u(x,t)dx = 0$ and thus

$$\int_{\mathbb{T}} u(x,t)dx = \int_{\mathbb{T}} u_0(x)dx.$$

Now set $v(x,t) = u(x-ct,t) - c$ and observe that if u solves KdV with initial data $u(x,0) = u_0$, then v solves

$$v_t + 2cv_x + v_{xxx} + vv_x = 0$$

with $v(x,0) = u_0(x) - c$. Since if we integrate in time we still have that $\partial_t \int_{\mathbb{T}} v(x,t)dx = 0$ we conclude that

$$\int_{\mathbb{T}} v(x,t)dx = \int_{\mathbb{T}} v_0(x)dx = \int_{\mathbb{T}} u_0(x)dx - 2\pi c.$$

But now we can pick the constant c in such away that v has mean zero,

$$\int_{\mathbb{T}} v(x,t)dx = 0.$$

Of course v doesn't solve the original KdV anymore but the methods we are developing apply to the new equation step by step. The only differenence is that now the multiplier of the linear group is $k^3 - 2ck$ instead k^3. Notice that in all calculations that follow this replacement changes nothing. We start with the Strichartz estimates on the torus.

Theorem 5.18.

$$\textit{i)}\ \|W_t g\|_{L^4_{x,t\in\mathbb{T}}} \lesssim \|g\|_{L^2},$$
$$\textit{ii)}\ \|W_t g\|_{L^6_{x,t\in\mathbb{T}}} \lesssim \|g\|_{H^\epsilon}, \quad \textit{for any } \epsilon > 0.$$

Proof. We will only prove ii). The proof of i) is simpler, see the remark below.

First assume that $\widehat{g} = 0$ outside $[-N,N]$. We write

$$\|W_t g\|_{L^6_{x,t\in\mathbb{T}}} = \sum_{\substack{k_1,k_2,k_3\in[-N,N]\\ j_1,j_2,j_3\in[-N,N]}} \widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)\overline{\widehat{g}(j_1)\widehat{g}(j_2)\widehat{g}(j_3)}$$
$$\times \int_{\mathbb{T}^2} e^{it(k_1^3+k_2^3+k_3^3-j_1^3-j_2^3-j_3^3)+ix(k_1+k_2+k_3-j_1-j_2-j_3)}dtdx.$$

Performing the integration in x,t, we obtain

$$\|W_t g\|^6_{L^6_{x,t\in\mathbb{T}}} = (2\pi)^2 \sum_{\substack{k_1^3+k_2^3+k_3^3=j_1^3+j_2^3+j_3^3\\ k_1+k_2+k_3=j_1+j_2+j_3}} \widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)\overline{\widehat{g}(j_1)\widehat{g}(j_2)\widehat{g}(j_3)}$$
$$= (2\pi)^2 \sum_{p,q} \sum_{\substack{(k_1,k_2,k_3)\in A_{p,q}\\ (j_1,j_2,j_3)\in A_{p,q}}} \widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)\overline{\widehat{g}(j_1)\widehat{g}(j_2)\widehat{g}(j_3)}$$
$$= (2\pi)^2 \sum_{p,q} \Big| \sum_{(k_1,k_2,k_3)\in A_{p,q}} \widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)\Big|^2,$$

where

$$A_{p,q} = \{(k_1, k_2, k_3) \in [-N, N]^3 : k_1 + k_2 + k_3 = p, k_1^3 + k_2^3 + k_3^3 = q\}.$$

We claim that for any $\epsilon > 0$, $\# A_{p,q} \lesssim N^\epsilon$. Indeed, writing

$$q - p^3 = k_1^3 + k_2^3 + k_3^3 - (k_1 + k_2 + k_3)^3 = -3(k_1 + k_2)(k_2 + k_3)(k_1 + k_3),$$

we see that quantities $k_i + k_j$ can take at most N^ϵ different values by using the standard fact that the number of divisors of an integer N is at most N^ϵ for any $\epsilon > 0$. Since the quantities $k_i + k_j$ uniquely determine k_1, k_2, k_3, we are done.

Using the claim and Cauchy-Schwarz, we see that

$$\begin{aligned}
\|W_t g\|_{L^6_{x,t\in\mathbb{T}}} &\lesssim \sum_{p,q} \sum_{(k_1,k_2,k_3)\in A_{p,q}} |\widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)|^2 \sum_{(k_1',k_2',k_3')\in A_{p,q}} 1 \\
&\lesssim N^\epsilon \sum_{p,q} \sum_{(k_1,k_2,k_3)\in A_{p,q}} |\widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)|^2 \\
&= N^\epsilon \sum_{(k_1,k_2,k_3)\in[-N,N]^3} |\widehat{g}(k_1)\widehat{g}(k_2)\widehat{g}(k_3)|^2 = N^\epsilon \|g\|_{L^2}^6.
\end{aligned}$$

For arbitrary $g \in H^\epsilon$, we write

$$g = \widehat{g}(0) + \sum_{n=0} g_n,$$

where $\widehat{g_n}(j) = \chi_{[2^n,2^{n+1})}(|j|)\widehat{g}(j)$. Taking the $L^6_{\mathbb{T}^2}$ norm of $W_t g$ and using the inequality above yields the statement.

Remark 5.19. *In the case of L^4 norm the $A_{p,q}$ set is defined as*

$$A_{p,q} = \{(k_1, k_2) \in [-N, N]^2 : k_1 + k_2 = p, k_1^3 + k_2^3 = q\}.$$

Note that this set has cardinality at most 4.

$\square$

Corollary 5.20. *For any space-time function u and $b > 1/2$, we have*

i) $\|u\|_{L^4_{x,t\in\mathbb{T}}} \lesssim \|u\|_{X^{0,b}}$,

ii) $\|u\|_{L^6_{x,t\in\mathbb{T}}} \lesssim \|u\|_{X^{\epsilon,b}}$, *for any* $\epsilon > 0$.

Proof. By Fourier inversion and then a change of variable we have

$$\begin{aligned}
u(t,x) &= \int\int \widehat{u}(\xi,\tau) e^{it\tau + ix\xi} d\xi d\tau \\
&= \int\int e^{it\xi^3} \widehat{u}(\xi, \tau + \xi^3) e^{it\tau + ix\xi} d\xi d\tau \\
&= \int e^{it\tau} W_t \psi_\tau d\tau,
\end{aligned}$$

where $\widehat{\psi_\tau}(\xi) = \widehat{u}(\xi, \tau + \xi^3)$. Therefore,

$$\begin{aligned}
\|u\|_{L^4_{x,t\in\mathbb{T}}} &\le \int \|W_t \psi_\tau\|_{L^4_{x,t\in\mathbb{T}}} d\tau \lesssim \int \|\psi_\tau\|_{L^2} d\tau \\
&= \int \left\|\widehat{u}(\xi, \tau + \xi^3)\right\|_{L^2_\xi} d\tau \\
&\le \|\langle\tau\rangle^{-b}\|_{L^2_\tau} \left\|\widehat{u}(\xi, \tau + \xi^3)\langle\tau\rangle^b\right\|_{L^2_\xi L^2_\tau} \\
&\lesssim \|u\|_{X^{0,b}}.
\end{aligned}$$

The proof of ii) is similar. □

We now present Bourgain's refinement of the L^4 Strichartz estimate. This refinement allows us to extract a power of δ in the well-posedness proof.

Theorem 5.21. *For any space-time function u*

$$\|u\|_{L^4_{x\in\mathbb{T}, t\in\mathbb{R}}} \lesssim \|u\|_{X^{0,1/3}}.$$

Proof. Let $u = \sum_{m=0}^{\infty} u_{2^m}$, where $\widehat{u_{2^m}} = \widehat{u}\chi_{2^m \le \langle\tau - k^3\rangle < 2^{m+1}}$. Note that by Plancherel

$$\|u\|^2_{X^{0,1/3}} \approx \sum_{m=0}^{\infty} 2^{2m/3} \|u_{2^m}\|^2_{L^2_{x,t}}. \tag{5.13}$$

We write

$$\|u\|^2_{L^4_{x,t}} = \|u^2\|_{L^2_{x,t}} \le 2 \sum_{m\le m'} \|u_{2^m} u_{2^{m'}}\|_{L^2} = 2 \sum_{m,n\ge 0} \|u_{2^m} u_{2^{m+n}}\|_{L^2}.$$

We will estimate

$$\begin{aligned}
\|u_{2^m} u_{2^{m+n}}\|_{L^2} &= \|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_{k,\tau}} \\
&= \Big\| \sum_{k_1} \int \widehat{u_{2^m}}(k_1, \tau_1) \widehat{u_{2^{m+n}}}(k - k_1, \tau - \tau_1) d\tau_1 \Big\|_{L^2_{k,\tau}}
\end{aligned} \tag{5.14}$$

separately in the range $|k| \le 2^a$ and $|k| > 2^a$.

In the former case, for each $|k| \le 2^a$, we put the L^2_τ norm inside the sum and apply Young's inequality to obtain

$$\begin{aligned}
\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_\tau} &\le \sum_{k_1} \Big\| \int \widehat{u_{2^m}}(k_1, \tau_1) \widehat{u_{2^{m+n}}}(k - k_1, \tau - \tau_1) d\tau_1 \Big\|_{L^2_\tau} \\
&\le \sum_{k_1} \|\widehat{u_{2^m}}(k_1, \cdot)\|_{L^1} \|\widehat{u_{2^{m+n}}}(k - k_1, \cdot)\|_{L^2} \\
&\lesssim 2^{m/2} \sum_{k_1} \|\widehat{u_{2^m}}(k_1, \cdot)\|_{L^2} \|\widehat{u_{2^{m+n}}}(k - k_1, \cdot)\|_{L^2} \\
&\le 2^{m/2} \Big(\sum_{k_1} \|\widehat{u_{2^m}}(k_1, \cdot)\|^2_{L^2} \Big)^{1/2} \Big(\sum_{k_1} \|\widehat{u_{2^{m+n}}}(k - k_1, \cdot)\|^2_{L^2} \Big)^{1/2} \\
&= 2^{m/2} \|u_{2^m}\|_{L^2_{x,t}} \|u_{2^{m+n}}\|_{L^2_{x,t}}.
\end{aligned}$$

Therefore, taking the $L^2_{|k|\le 2^a}$ norm we obtain

$$\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_{|k|\le 2^a,\tau}} \lesssim 2^{\frac{a+m}{2}} \|u_{2^m}\|_{L^2_{x,t}} \|u_{2^{m+n}}\|_{L^2_{x,t}}. \tag{5.15}$$

In the latter case, we have

$$\begin{aligned}
&\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_{|k|>2^a,\tau}} \\
&\le \Big\| \Big(\sum_{k_1} \int |\widehat{u_{2^m}}(k_1,\tau_1)|^2 |\widehat{u_{2^{m+n}}}(k-k_1,\tau-\tau_1)|^2 d\tau_1 \Big)^{1/2} \big(\chi_m * \chi_{m+n}(k,\tau)\big)^{1/2} \Big\|_{L^2_{|k|>2^a,\tau}},
\end{aligned}$$

where $\chi_m(k,\tau) = \chi_{2^m \le \langle \tau - k^3\rangle < 2^{m+1}}$. Taking the supremum of the convolution outside the norm we obtain

$$\begin{aligned}
&\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_{|k|>2^a,\tau}} \\
&\le \|\chi_m * \chi_{m+n}\|^{1/2}_{L^\infty_{|k|>2^a,\tau}} \Big\| \Big(\sum_{k_1} \int |\widehat{u_{2^m}}(k_1,\tau_1)|^2 |\widehat{u_{2^{m+n}}}(k-k_1,\tau-\tau_1)|^2 d\tau_1 \Big)^{1/2} \Big\|_{L^2_{k,\tau}} \\
&= \|\chi_m * \chi_{m+n}\|^{1/2}_{L^\infty_{|k|>2^a,\tau}} \|u_{2^m}\|_{L^2_{x,t}} \|u_{2^{m+n}}\|_{L^2_{x,t}}.
\end{aligned}$$

To estimate the convolution, we write for fixed $|k| > 2^a$ and τ,

$$\chi_m * \chi_{m+n}(k,\tau) = \sum_{k_1} \int \chi_m(k_1,\tau_1)\chi_{m+n}(k_1,\tau-\tau_1) d\tau_1.$$

By the support condition on χ_m and χ_{m+n}, we have

$$\tau_1 = k_1^3 + O(2^m), \quad \tau - \tau_1 = (k-k_1)^3 + O(2^{m+n}).$$

Therefore for each fixed k_1, the τ_1 integral is $O(2^m)$. To calculate the number of k_1s for which the integral is nonzero, note that

$$\tau = k_1^3 + (k-k_1)^3 + O(2^{m+n}) \implies k^2 - 3k_1 k + 3k_1^2 = \frac{\tau}{k} + O(2^{m+n-a}).$$

This implies that

$$3\big(k_1 - k/2\big)^2 = \frac{\tau}{k} - \frac{k^2}{4} + O(2^{m+n-a}).$$

Therefore, k_1 takes $O(2^{\frac{m+n-a}{2}})$ values. Using this bound, we obtain

$$\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_{|k|>2^a,\tau}} \lesssim 2^{\frac{3m+n-a}{4}} \|u_{2^m}\|_{L^2_{x,t}} \|u_{2^{m+n}}\|_{L^2_{x,t}}. \tag{5.16}$$

Combining (5.15) and (5.16), and choosing $a = \frac{m+n}{3}$, we obtain

$$\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_{k,\tau}} \lesssim 2^{\frac{4m+n}{6}} \|u_{2^m}\|_{L^2_{x,t}} \|u_{2^{m+n}}\|_{L^2_{x,t}}.$$

Finally we estimate

$$\begin{aligned}
\|u\|^2_{L^4_{x,t}} = \|u^2\|_{L^2_{x,t}} &\le 2 \sum_{m,n\ge 0} \|u_{2^m} u_{2^{m+n}}\|_{L^2} \\
&\lesssim \sum_{n\ge 0} 2^{-\frac{n}{6}} \sum_{m \ge 0} 2^{\frac{m}{3}} \|u_{2^m}\|_{L^2_{x,t}} 2^{\frac{m+n}{3}} \|u_{2^{m+n}}\|_{L^2_{x,t}}.
\end{aligned}$$

By the Cauchy-Schwarz inequality in the m sum and (5.13) gives

$$\|u\|^2_{L^4_{x,t}} \lesssim \|u\|^2_{X^{0,1/3}} \sum_{n\ge 0} 2^{-\frac{n}{6}} \lesssim \|u\|^2_{X^{0,1/3}}.$$

$\square$

Unfortunately, for the H^s local theory of KdV on the torus, the suitable space is $X^{s,1/2}$ (to kill the derivative one needs to take $b = 1/2$). This space does not embedd into $C(\mathbb{R}; H^s)$. Therefore we define the space Y^s via the norm

$$\|u\|_{Y^s} = \|u\|_{X^{s,1/2}} + \|\langle k\rangle^s \widehat{u}(k,\tau)\|_{\ell^2_k L^1_\tau}.$$

The restricted space Y^s_δ is defined accordingly.

Note that for each t,

$$\|u(\cdot,t)\|^2_{H^s} = \sum_k |(u(\widehat{k}),t)|^2 \langle k\rangle^{2s} = \sum_k \Big| \int_\tau \widehat{u}(\widehat{k},\tau) e^{it\tau} d\tau \Big|^2 \langle k\rangle^{2s} \leq \|u\|^2_{Y^s}.$$

Continuity in t follows by the dominated convergence theorem. Thus, Y^s embedds into $C(\mathbb{R}; H^s)$.

We will present suitable versions of the lemmas in the previous section for the Y^s norms. As in the previous section, we will ignore the δ dependence in the proofs.

Lemma 5.22. *For any s real,*

$$\|\eta(t) W_t g\|_{Y^s_\delta} \lesssim \|g\|_{H^s}.$$

Proof. It suffices to bound the second part of the Y^s norm.

$$\big\|\widehat{\eta W_t g}(k,\tau)\langle k\rangle^s\big\|_{\ell^2_k L^1_\tau} = \big\|\widehat{\eta}(\tau - k^3)\widehat{g}(k)\langle k\rangle^s\big\|_{\ell^2_k L^1_\tau} \lesssim \|g\|_{H^s}.$$

□

To estimate the Duhamel term we introduce the "dual" space

$$\|u\|_{Z^s} = \|u\|_{X^{s,-1/2}} + \|\widehat{u}(k,\tau)\langle k\rangle^s \langle \tau - k^3\rangle^{-1}\|_{\ell^2_k L^1_\tau}.$$

Again, Z^s_δ is defined accordingly.

Lemma 5.23. *We have*

$$\Big\|\eta(t)\int_0^t W_{t-s}F(s)ds\Big\|_{Y^s_\delta} \lesssim \|F\|_{Z^s_\delta}.$$

Proof. We first estimate the $X^{s,1/2}$ part of the norm:

$$\begin{aligned}\Big\|\eta(t)\int_0^t W_{t-s}F(s)ds\Big\|_{X^{s,1/2}} &= \Big\|\eta(t)\int_0^t W_{-s}F(s)ds\Big\|_{H^s_x H^b_t} \\ &= \Big\|\eta(t)\int_0^t \big[W_{-s}F(s)\big](\widehat{k})ds\langle k\rangle^s\Big\|_{\ell^2_k H^{1/2}_t}.\end{aligned}$$

Using (5.9),

$$\Big\|\eta(t)\int_0^t f(s)ds\Big\|_{H^{1/2}} \lesssim \|f\|_{H^{-1/2}} + \|\langle z\rangle^{-1}\widehat{f}\|_{L^1},$$

we estimate this by

$$\big\|\big[W_{-t}F(t)\big](\widehat{k})\langle k\rangle^s\big\|_{\ell^2_k H^{-1/2}_t} + \big\|\widehat{W_{-t}F}(t)(k,\tau)\langle k\rangle^s\langle\tau\rangle^{-1}\big\|_{\ell^2_k L^1_\tau} = \|F\|_{Z_s}.$$

To estimate the other part of the Y^s norm, define $\mathcal{D}(x,t) = \eta(t)\int_0^t W_{t-s}F(s)ds$. Recall that

$$\widehat{\mathcal{D}}(k,\tau) = \int \frac{\widehat{\eta}(\tau - z - k^3) - \widehat{\eta}(\tau - k^3)}{iz}\widehat{F}(z+k^3,k)dz.$$

Using this we estimate

$$\begin{aligned}\|\langle k\rangle^s \widehat{\mathcal{D}}(k,\tau)\|_{\ell^2_k L^1_\tau} &\leq \Big\| \int \Big\| \frac{\widehat{\eta}(\tau - z - k^3) - \widehat{\eta}(\tau - k^3)}{iz} \Big\|_{L^1_\tau} |\widehat{F}(z+k^3,k)|\langle k\rangle^s dz \Big\|_{\ell^2_k} \\ &\lesssim \Big\| \int \langle z\rangle^{-1} |\widehat{F}(z+k^3,k)|\langle k\rangle^s dz \Big\|_{\ell^2_k} \leq \|F\|_{Z^s}.\end{aligned}$$

We obtained the second line by considering the cases $|z|<1$ and $|z|>1$ separately. In the former case we used the mean value theorem. □

Theorem 5.24. *Assume that u is a space-time function of mean zero for each t, then for $s > -1/2$ we have*

$$\|\partial_x(u^2)\|_{Z^s} \lesssim \|u\|_{X^{s,1/2}}\|u\|_{X^{s,1/3}}.$$

Proof. We will give the proof only for the range $s \in (-1/2, 0]$. The proof is easier for $s > 0$. We start with the first part of the Z^s norm:

$$\begin{aligned}\|\partial_x(u^2)\|_{X^{s,-1/2}} &= \sup_{\|w\|_{X^{-s,1/2}}=1} \Big| \int w\partial_x u^2 dtdx \Big| \\ &= \sup_{\|w\|_{X^{-s,1/2}}=1} \Big| \sum_{k_1+k_2+k_3=0} \int_{\tau_1+\tau_2+\tau_3=0} k_3 \widehat{u}(k_1,\tau_1)\widehat{u}(k_2,\tau_2)\widehat{w}(k_3,\tau_3)\Big|.\end{aligned}$$

Using the notation

$$\begin{aligned}f_1(k,\tau) = f_2(k,\tau) &= |\widehat{u}(k,\tau)|\langle k\rangle^s\langle \tau - k^3\rangle^{1/2}, \\ f_3(k,\tau) &= |\widehat{w}(-k,-\tau)|\langle k\rangle^{-s}\langle \tau - k^3\rangle^{1/2},\end{aligned}$$

we estimate the right hand side by

$$\sum_{k_1+k_2+k_3=0} \int_{\tau_1+\tau_2+\tau_3=0} \frac{\langle k_3\rangle^{1+s} f_1(k_1,\tau_1)f_2(k_2,\tau_2)f_3(k_3,\tau_3)}{\langle k_1\rangle^s\langle k_2\rangle^s\langle \tau_1 - k_1^3\rangle^{1/2}\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle^{1/2}}. \tag{5.17}$$

Note that because of the mean zero assumption $k_j \neq 0$ in the sum above. We continue by estimating the multiplier, setting $s = -\rho \in [0, 1/2)$,

$$\frac{\langle k_1\rangle^\rho\langle k_2\rangle^\rho\langle k_3\rangle^{1-\rho}}{\langle \tau_1 - k_1^3\rangle^{1/2}\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle^{1/2}}.$$

Notice that

$$\tau_1 - k_1^3 + \tau_2 - k_2^3 + \tau_3 - k_3^3 = (k_1+k_2)^3 - k_1^3 - k_2^3 = 3k_1k_2(k_1+k_2) = -3k_1k_2k_3.$$

Therefore (using $k_j \neq 0$)

$$\max(\langle \tau_1 - k_1^3\rangle, \langle \tau_2 - k_2^3\rangle, \langle \tau_3 - k_3^3\rangle) \gtrsim \langle k_1\rangle\langle k_2\rangle\langle k_3\rangle. \tag{5.18}$$

Assume that the largest one is $\langle \tau_1 - k_1^3\rangle$, the other cases are similar. The multiplier is estimated by (using $k_3 = -k_1 - k_2$)

$$\frac{\langle k_3\rangle^{\frac12 - \rho}}{\langle k_1\rangle^{\frac12-\rho}\langle k_2\rangle^{\frac12-\rho}\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle^{1/2}} \lesssim \frac{1}{\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle^{1/2}}.$$

Using this in (5.17), we obtain

$$\begin{aligned}(5.17) &\lesssim \sum_{k_1+k_2+k_3=0} \int_{\tau_1+\tau_2+\tau_3=0} \frac{f_1(k_1,\tau_1)f_2(k_2,\tau_2)f_3(k_3,\tau_3)}{\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle^{1/2}} \\ &= \int_{\mathbb{T}\times\mathbb{R}} \mathcal{F}^{-1}(f_1)\mathcal{F}^{-1}\Big(\frac{f_2}{\langle \tau - k^3\rangle^{1/2}}\Big)\mathcal{F}\Big(\frac{f_3}{\langle \tau - k^3\rangle^{1/2}}\Big)dxdt.\end{aligned}$$

Here we used Fourier multiplicatioin formula and the convolution structure. By using Hölder and then Theorem 5.21, we estimate this by

$$\begin{aligned}&\|\mathcal{F}^{-1}(f_1)\|_{L^2_{x,t}}\Big\|\mathcal{F}^{-1}\Big(\frac{f_2}{\langle \tau - k^3\rangle^{1/2}}\Big)\Big\|_{L^4_{x,t}}\Big\|\mathcal{F}\Big(\frac{f_3}{\langle \tau - k^3\rangle^{1/2}}\Big)\Big\|_{L^4_{x,t}} \\ &\lesssim \|f_1\|_{L^2}\Big\|\mathcal{F}^{-1}\Big(\frac{f_2}{\langle \tau - k^3\rangle^{1/2}}\Big)\Big\|_{X^{0,1/3}}\Big\|\mathcal{F}\Big(\frac{f_3}{\langle \tau - k^3\rangle^{1/2}}\Big)\Big\|_{X^{0,1/3}} \\ &= \|u\|_{X^{s,1/2}}\|u\|_{X^{s,1/3}}\|w\|_{X^{-s,1/3}} \leq \|u\|_{X^{s,1/2}}\|u\|_{X^{s,1/3}}\|w\|_{X^{-s,1/2}}.\end{aligned}$$

We continue with the second part of the Z^s norm. Using duality we write

$$\begin{aligned}&\Big\|\frac{\langle k\rangle^s \widehat{\partial_x u^2}(k,\tau)}{\langle \tau - k^3\rangle}\Big\|_{\ell^2_k L^1_\tau} \\ &\leq \sup_{\|w\|_{\ell^2_k L^\infty_\tau}=1} \sum_{k_1+k_2+k_3=0}\int_{\tau_1+\tau_2+\tau_3=0} \frac{\langle k_3\rangle^{1+s}|\widehat{u}(k_1,\tau_1)||\widehat{u}(k_2,\tau_2)||w(k_3,\tau_3)|}{\langle \tau_3 - k_3^3\rangle} \\ &= \sup_{\|w\|_{\ell^2_k L^\infty_\tau}=1} \sum_{k_1+k_2+k_3=0}\int_{\tau_1+\tau_2+\tau_3=0} \frac{\langle k_3\rangle^{1+s} f_1(k_1,\tau_1)f_2(k_2,\tau_2)|w(k_3,\tau_3)|}{\langle k_1\rangle^s\langle k_2\rangle^s\langle \tau_1 - k_1^3\rangle^{1/2}\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle},\end{aligned}$$

with f_1 and f_2 as above. By symmetry, we have two cases to consider.

Case 1) $\max\big(\langle \tau_1 - k_1^3\rangle, \langle \tau_2 - k_2^3\rangle, \langle \tau_3 - k_3^3\rangle\big) = \langle \tau_1 - k_1^3\rangle$.

Using (5.18), the multiplier is bounded by

$$\frac{\langle k_3\rangle^{\frac{1}{2}+s}}{\langle k_1\rangle^{\frac{1}{2}+s}\langle k_2\rangle^{\frac{1}{2}+s}\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle} \lesssim \frac{1}{\langle \tau_2 - k_2^3\rangle^{1/2}\langle \tau_3 - k_3^3\rangle}.$$

Using this bound as above we estimate norm in this case by

$$\begin{aligned}&\sup_{\|w\|_{\ell^2_k L^\infty_\tau}=1} \|f_1\|_{L^2}\Big\|\mathcal{F}^{-1}\Big(\frac{f_2}{\langle \tau - k^3\rangle^{1/2}}\Big)\Big\|_{X^{0,1/3}}\Big\|\mathcal{F}\Big(\frac{|w|}{\langle \tau - k^3\rangle}\Big)\Big\|_{X^{0,1/3}} \\ &= \|u\|_{X^{s,1/2}}\|u\|_{X^{s,1/3}} \sup_{\|w\|_{\ell^2_k L^\infty_\tau}=1}\Big\|\frac{w}{\langle \tau - k^3\rangle^{2/3}}\Big\|_{\ell^2_k L^2_\tau} \\ &\lesssim \|u\|_{X^{s,1/2}}\|u\|_{X^{s,1/3}}.\end{aligned}$$

In the last line we used Hölder's inequality in the τ variable.

Case 2) $\max\big(\langle \tau_1 - k_1^3\rangle, \langle \tau_2 - k_2^3\rangle, \langle \tau_3 - k_3^3\rangle\big) = \langle \tau_3 - k_3^3\rangle$.

Using (5.18), we estimate

$$\langle \tau_3 - k_3^3\rangle \gtrsim \langle k_1\rangle\langle k_2\rangle\langle k_3\rangle \gtrsim \langle k_3\rangle^2.$$

Therefore we have

$$\langle \tau_3 - k_3^3\rangle = \langle \tau_3 - k_3^3\rangle^{-s}\langle \tau_3 - k_3^3\rangle^{1+s} \gtrsim \langle k_1\rangle^{-s}\langle k_2\rangle^{-s}\langle k_3\rangle^{-s}\big(\langle \tau_3 - k_3^3\rangle + \langle k_3\rangle^2\big)^{1+s}.$$

Using this, we estimate the multiplier by

$$\frac{\langle k_3\rangle^{1+2s}}{\langle\tau_1-k_1^3\rangle^{1/2}\langle\tau_2-k_2^3\rangle^{1/2}\big(\langle\tau_3-k_3^3\rangle+\langle k_3\rangle^2\big)^{1+s}}.$$

Using this as above (switching the roles of f_1 and w), we bound the norm by

$$\Big\|\mathcal{F}^{-1}\Big(\frac{f_1}{\langle\tau-k^3\rangle^{1/2}}\Big)\Big\|_{X^{0,1/3}}\Big\|\mathcal{F}^{-1}\Big(\frac{f_2}{\langle\tau-k^3\rangle^{1/2}}\Big)\Big\|_{X^{0,1/3}}\sup_{\|w\|_{\ell^2_k L^\infty_\tau}=1}\Big\|\frac{w(k,\tau)\langle k\rangle^{1+2s}}{\big(\langle\tau-k^3\rangle+\langle k\rangle^2\big)^{1+s}}\Big\|_{L^2_{k,\tau}}$$

$$\lesssim \|u\|^2_{X^{s,1/3}}\Big\|\frac{\langle k\rangle^{1+2s}}{\big(\langle\tau-k^3\rangle+\langle k\rangle^2\big)^{1+s}}\Big\|_{\ell^\infty_k L^2_\tau}$$

$$\lesssim \|u\|_{X^{s,1/2}}\|u\|_{X^{s,1/3}}.$$

The last line follows from the inequality (using $s>-1/2$)

$$\int\frac{1}{\big(|\tau|+\langle k\rangle^2\big)^{2+2s}}d\tau\lesssim\frac{1}{\langle k\rangle^{2(1+2s)}}.$$

□

Corollary 5.25. *Let $\delta\in(0,1)$. Assume that u is a space-time function of mean zero for each t, then for $s>-1/2$ we have*

$$\|\partial_x(u^2)\|_{Z^s_\delta}\lesssim\delta^{\frac16-}\|u\|^2_{X^{s,1/2}_\delta}.$$

We are ready to run the contraction argument. Using Lemma 5.22, Lemma 5.23, and Corollary 5.25, we have

$$\|\Phi u\|_{Y^s}\lesssim\|u_0\|_{H^s}+\delta^{\frac16-}\|u\|^2_{X^{s,1/2}_\delta}.$$

Thus, one can close the contraction in the space (with $M=M(\|u_0\|_{H^s})$ and $\delta=\delta(M)$)

$$X=\{u:\|u\|_{Y^s_\delta}\le M\}.$$

5.4. **Differentiation by parts method on $\mathbb{T}$.** We pressent below an alternative method for proving local and global well-posedness for L^2 data on $\mathbb{T}$. The method can be summarized as changing variables and differentiating by parts in the time variable. This eliminates the derivative in the nonlinearity by replacing it with a higher order pure power nonlinearity. One has to be careful with the resonant terms and do the differentiation by parts twice for this method to work. Moreover to close the contraction we will consider high and low frequencies separately. As in the previous section we will work with mean zero initial data. The idea of what follows is essentially in [2]. See also [25].

Using the Fourier series representation

$$u(x,t)=\sum_{k\in\mathbb{Z}_0}u_k(t)e^{ikx}$$

with

$$u_k:=\widehat{u}(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(t,x)e^{-ikx}dx$$

write KdV,

$$u_t = u_{xxx} + uu_x,$$

on the Fourier side as

$$\partial_t u_k = \frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2} - ik^3 u_k.$$

Then, using the identity

$$(k_1+k_2)^3 - k_1^3 - k_2^3 = 3(k_1+k_2)k_1k_2,$$

and the transformation

$$v_k(t) = u_k(t)e^{ik^3t}$$

the equation can be written in the form

$$\partial_t v_k = \frac{ik}{2} \sum_{k_1+k_2=k} e^{i3kk_1k_2t} v_{k_1} v_{k_2}. \tag{5.19}$$

Integrating both sides in t, we have

$$v_k(t) - v_k(0) = \int_0^t \frac{ik}{2} \sum_{k_1+k_2=k} e^{i3kk_1k_2s} v_{k_1} v_{k_2} ds. \tag{5.20}$$

From now on we say u is a strong solution of KdV on $[-\delta,\delta]$ if $u \in C([-\delta,\delta]; L^2(\mathbb{T}))$ and if for each $k \in \mathbb{Z}$, $t \in [-\delta,\delta]$, $v_k(t) = u_k(t)e^{ik^3t}$ satisfies (5.20).

Remarks. i) Solutions as defined above also satisfy (5.19) for each k and t. Moreover, we have the following bound

$$\sup_{t\in[-\delta,\delta]} |\partial_t v_k| \lesssim |k|$$

with the implicit constant depending only on $\|v\|_{L^\infty_{[-\delta,\delta]} L^2_{\mathbb{T}}}$.

ii) Below, we will perform the differentiation by parts process. For any given solution, v, the bound on $\partial_t v_k$ suffices to justify this process. Therefore any given solution is also a solution of the integral equation (5.25) below. Indeed, it suffices to check that for each k, one can change the order of sum and differentiation, which follows from the bound above and the mean value theorem.

Since $e^{i3kk_1k_2t} = \partial_t(\frac{1}{3ikk_1k_2} e^{i3kk_1k_2t})$ differentiation by parts and (5.19) yields

$$\partial_t v_k = \partial_t\Big(\frac{1}{2} ik \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t} v_{k_1} v_{k_2}}{3ikk_1k_2}\Big) - \frac{1}{2} ik \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{3ikk_1k_2} \partial_t(v_{k_1} v_{k_2})$$

$$= \frac{1}{6}\partial_t\Big(\sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t} v_{k_1} v_{k_2}}{k_1k_2}\Big) - \frac{1}{6} \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2} (\partial_t v_{k_1} v_{k_2} + \partial_t v_{k_2} v_{k_1}).$$

Note that since $v_0 = 0$, the terms corresponding to $k_1 = 0$ or $k_2 = 0$ are not actually present in the above sums. The last two terms are symmetric with respect to k_1 and k_2 and thus we can consider only one of them. Using (5.19) we have

$$\sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2} v_{k_1} \partial_t v_{k_2} = \frac{i}{2} \sum_{k=k_1+k_2} \frac{e^{3ikk_1k_2t}}{k_1} v_{k_1} \Big(\sum_{\mu+\lambda=k_2} e^{3itk_2\mu\lambda} v_\mu v_\lambda\Big)$$

$$= \frac{i}{2} \sum_{k=k_1+\mu+\lambda} \frac{v_{k_1} v_\mu v_\lambda}{k_1} e^{3it[kk_1(\mu+\lambda)+\mu\lambda(\mu+\lambda)]}.$$

We note that $\mu+\lambda$ can not be zero since $\mu+\lambda=k_2$. Using the identity

$$kk_1 + \mu\lambda = (k_1+\mu+\lambda)k_1 + \mu\lambda = (k_1+\mu)(k_1+\lambda)$$

and thus by renaming the variables $k_2=\mu, k_3=\lambda$, we have that

$$\sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2} v_{k_1} \partial_t v_{k_2} = \frac{i}{2} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3}.$$

All in all we have that

$$\partial_t \left(v_k - \frac{1}{6} B_2(v,v)_k \right) = -\frac{i}{6} R_3(v,v,v)_k$$

where

$$B_2(u,v)_k = \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t} u_{k_1} v_{k_2}}{k_1k_2}$$

and

$$R_3(u,v,w)_k = \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} u_{k_1} v_{k_2} w_{k_3}.$$

Now let's single out the resonant terms for which

$$(k_1+k_2)(k_3+k_1) = 0 \tag{5.21}$$

and write

$$R_3(v,v,v)_k = R_{3r}(v,v,v)_k + R_{3nr}(v,v,v)_k$$

where the subscript r and nr stands for the resonant and non-resonant terms respectively. Thus,

$$R_{3r}(v,v,v)_k = \sum^{\mathrm{r}}_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{v_{k_1} v_{k_2} v_{k_3}}{k_1}$$

and

$$R_{3nr}(v,v,v)_k = \sum^{\mathrm{nr}}_{k_1+k_2+k_3=k} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3},$$

where $\sum^{\mathrm{nr}}$ means that the sum contains only the terms with non-zero exponents. Similarly, $\sum^{\mathrm{r}}$ means that the sum contains only the terms with zero exponents. The set for which (5.21) holds is the disjoint union of the following 3 sets

$$S_1 = \{k_1+k_2=0\} \cap \{k_3+k_1=0\} \Leftrightarrow \{k_1=-k,\ k_2=k,\ k_3=k\},$$

$$S_2 = \{k_1+k_2=0\} \cap \{k_3+k_1\neq 0\} \Leftrightarrow \{k_1=j,\ k_2=-j,\ k_3=k,\ |j|\neq|k|\},$$

$$S_3 = \{k_3+k_1=0\} \cap \{k_1+k_2\neq 0\}\} \Leftrightarrow \{k_1=j,\ k_2=k,\ k_3=-j,\ |j|\neq|k|\}.$$

Thus

$$R_{3r}(v,v,v)_k = \sum_{\lambda=1}^{3} \sum_{S_\lambda} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1} = \frac{v_{-k}v_kv_k}{-k} + v_k \sum_{\substack{j\in\mathbb{Z}_0 \\ |j|\neq|k|}} \frac{v_jv_{-j}}{j} + v_k \sum_{\substack{j\in\mathbb{Z}_0 \\ |j|\neq|k|}} \frac{v_jv_{-j}}{j}.$$

Note that the second and third terms in the sum above are identically zero due to the symmetry relation $j \leftrightarrow -j$. Thus

$$R_{3r}(v,v,v)_k = -\frac{v_k}{k}|v_k|^2,$$

where we used $v_{-k} = \overline{v}_k$. We obtain

$$\partial_t \left(v_k - \frac{1}{6} B_2(v,v)_k \right) = \frac{i}{6k} v_k |v_k|^2 - \frac{i}{6} R_{3nr}(v,v,v)_k.$$

Since the exponent in the last term is not zero we can differentiate by parts one more time and obtain that

$$R_{3nr}(v,v,v)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3} =$$

$$\frac{1}{3i}\partial_t B_3(v,v,v)_k - \frac{1}{3i} \sum_{k_1+k_2+k_3=k}^{\mathrm{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \times$$

$$(\partial_t v_{k_1} v_{k_2} v_{k_3} + \partial_t v_{k_2} v_{k_1} v_{k_3} + \partial_t v_{k_3} v_{k_1} v_{k_2})$$

where

$$B_3(u,v,w)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} u_{k_1} v_{k_2} w_{k_3}.$$

As before we express time derivatives using (5.19). The terms containing $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ produce the same expressions and a calculation reveals that

$$\sum_{k_1+k_2+k_3=k}^{\mathrm{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \times$$

$$(\partial_t v_{k_1} v_{k_2} v_{k_3} + \partial_t v_{k_2} v_{k_1} v_{k_3} + \partial_t v_{k_3} v_{k_1} v_{k_2}) = iB_4(v,v,v,v)_k$$

where

$$B_4(u,v,w,z)_k = \frac{1}{2} B_4^1(u,v,w,z)_k + B_4^2(u,v,w,z)_k.$$

From now on $\sum^*$ means that the sum is over all indices for which the denominator do not vanish.The term corresponding to $\partial_t v_{k_1}$ is

$$B_4^1(u,v,w,z)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\star} \frac{e^{it\psi(k_1,k_2,k_3,k_4)}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1} v_{k_2} w_{k_3} z_{k_4},$$

and the sum of the terms corresponding to $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ is

$$B_4^2(u,v,w,z)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\star} \frac{e^{it\psi(k_1,k_2,k_3,k_4)}(k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1} v_{k_2} w_{k_3} z_{k_4}.$$

The phase function ψ will be irrelevant for our calculations since it is going to be estimated out by taking absolute values inside the sums. Hence for $R_{3nr}(v,v,v)_k$ we have:

$$R_{3nr}(v,v,v)_k = \frac{1}{3i}\partial_t B_3(v,v,v)_k - \frac{1}{3}\left(\frac{1}{2} B_4^1(v,v,v,v)_k + B_4^2(v,v,v,v)_k \right).$$

If we put everything together and combining the two B_4 terms in one we obtain

$$\partial_t \big(v_k - B(v) \big) = \frac{i v_k |v_k|^2}{6k} + \frac{i}{18} B_4(v)_k, \tag{5.22}$$

where

$$B(v)_k = -\frac{1}{6}\sum_{k_1+k_2=k}\frac{e^{i3kk_1k_2t}v_{k_1}v_{k_2}}{k_1k_2}+\frac{1}{18}\sum^*_{k_1+k_2+k_3=k}\frac{e^{i3(k_1+k_2)(k_1+k_3)(k_2+k_3)t}v_{k_1}v_{k_2}v_{k_3}}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)}$$

$$B_4(v)_k = \frac{1}{2}\sum^*_{k_1+k_2+k_3+k_4=k}\frac{e^{i\psi(k_1,k_2,k_3,k_4)t}(2k_3+2k_4+k_1)v_{k_1}v_{k_2}v_{k_3}v_{k_4}}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}.$$

Integrating (5.22) from 0 to t, we obtain

$$v_k(t) = v_k(0) + B(v)(t) - B(v)(0) + \int_0^t \Big(\frac{iv_k|v_k|^2}{6k} + \frac{i}{18}B_4(v)_k\Big)(s)ds. \tag{5.23}$$

Note that if we integrate the original equation (5.19), we obtain

$$v_k(t) = v_k(0) + \int_0^t \frac{ik}{2}\sum_{k_1+k_2=k} e^{i3kk_1k_2s}v_{k_1}v_{k_2}ds. \tag{5.24}$$

Fix N large to be determined later. Define the operator T as follows

$$T(v)_k(t) = \begin{cases} v_k(0) + B(v)(t) - B(v)(0) + \int_0^t \big(\frac{iv_k|v_k|^2}{6k} + \frac{i}{18}B_4(v)_k\big)(s)ds & |k|>N \\ v_k(0) + \int_0^t \frac{ik}{2}\sum_{k_1+k_2=k} e^{i3kk_1k_2s}v_{k_1}v_{k_2}ds & |k|\le N \end{cases} \tag{5.25}$$

Proposition 5.26.

$$\|B(v)\|_{\ell^2_{|k|>N}} \lesssim \frac{1}{N^{1/4}}\big(\|v\|^2_{\ell^2} + \|v\|^3_{\ell^2}\big) \tag{5.26}$$

$$\|B_4(v)\|_{\ell^2_{|k|>N}} \lesssim \|v\|^4_{\ell^2} \tag{5.27}$$

$$\Big\|\frac{v_k|v_k|^2}{k}\Big\|_{\ell^2_{|k|>N}} \lesssim \frac{1}{N}\|v\|^3_{\ell^2} \tag{5.28}$$

$$\Big\|k\sum_{k_1+k_2=k} e^{i3kk_1k_2s}v_{k_1}v_{k_2}\Big\|_{\ell^2_{|k|\le N}} \lesssim N^{3/2}\|v\|^2_{\ell^2} \tag{5.29}$$

Using this proposition, we now prove that T is a contraction on

$$X = \{v\in C([-\delta,\delta];\ell^2) : \|v\|_{L^\infty_{[-\delta,\delta]}\ell^2} \le M\},$$

where N, M, δ depends on $\|u_0\|_{L^2}$. Indeed,

$$\|Tv\|_{L^\infty_{[-\delta,\delta]}\ell^2} \lesssim \|v(0)\|_{L^2} + \frac{1}{N^{1/4}}\big(\|v\|^2_{L^\infty_{[-\delta,\delta]}\ell^2} + \|v\|^3_{L^\infty_{[-\delta,\delta]}\ell^2}\big)$$
$$+ \delta\frac{1}{N}\|v\|^3_{L^\infty_{[-\delta,\delta]}\ell^2} + \delta\|v\|^4_{L^\infty_{[-\delta,\delta]}\ell^2} + \delta N^{3/2}\|v\|^2_{L^\infty_{[-\delta,\delta]}\ell^2}$$

First choosing M large depending on $\|u_0\|_{L^2}$, then N large depending on M, and finally δ small depending on N and M, we see that T is a contraction on X. This gives us a unique solution in X and continuous dependence on initial data for the equation $Tv = v$. Note that the smooth solutions of KdV, which exists by the Bona-Smith method presented above, also solves this equation by the remark in the beginning of this section.

Given L^2 initial data, $v(0)$, we need to prove that the solution, the fixed point v of T, also solves KdV. To do this, we approximate $v(0)$ by a smooth sequence, $v_n(0)$, and obtain the corresponding solutions v_n of KdV. Since v_n also solve the new equation, by continuous dependence on initial data v_n converges to v in $C([-\delta,\delta];\ell^2)$. Therefore, for each fixed k, taking the limit as $n\to\infty$ in (5.24), we see that v also satisfies (5.24), and is a solution of KdV.

We now prove the uniqueness of the solution of KdV for a given initial data in L^2. Let $v_1, v_2 \in C([-\delta,\delta];L^2(\mathbb{T}))$ be two solutions of KdV with the same initial data. By the remark in the beginning of this section, v_1 and v_2 are fixed points of the equation $Tv = v$. Therefore, $v_1 = v_2$.

Remark 5.27. *Uniqueness as it is proved above is known as "unconditional uniqueness" in the literature. Note that the methods used in the previous two sections give uniquness only in a proper subset of* $C([-\delta,\delta],H^s)$.

We now prove Proposition 5.26.

Proof of Proposition 5.26. We start with (5.26). For $|k|>N$, we have

$$|B(v)_k| \lesssim \frac{1}{N}\sum_{k_1+k_2=k}\frac{|v_{k_1}||v_{k_2}|}{|k_2|} + \frac{1}{N^{1/4}}\sideset{}{^*}\sum_{k_1+k_2+k_3=k}\frac{|v_{k_1}||v_{k_2}||v_{k_3}|}{|k_1||k_2|^{3/4}}.$$

The first one follows assuming by symmetry that $|k_1|\gtrsim|k|$, while the second follows using

$$|(k_1+k_2)(k_1+k_3)(k_2+k_3)| \gtrsim \max(|k_1|,|k_2|,|k_3|) \gtrsim |k_2|^{3/4}|k|^{1/4}.$$

Taking the ℓ^2 norm, we have

$$\begin{aligned}\|B(v)\|_{\ell^2_{|k|>N}} &\lesssim \frac{1}{N}\Big\||v_k| * \frac{|v_k|}{|k|}\Big\|_{\ell^2} + \frac{1}{N^{1/4}}\Big\|\frac{|v_k|}{|k|} * \frac{|v_k|}{|k|^{3/4}} * |v_k|\Big\|_{\ell^2}\\ &\lesssim \frac{1}{N}\|v\|_{\ell^2}\Big\|\frac{|v_k|}{|k|}\Big\|_{\ell^1} + \frac{1}{N^{1/4}}\|v\|_{\ell^2}\Big\|\frac{|v_k|}{|k|}\Big\|_{\ell^1}\Big\|\frac{|v_k|}{|k|^{3/4}}\Big\|_{\ell^1}\\ &\lesssim \frac{1}{N^{1/4}}\big(\|v\|_{\ell^2}^2 + \|v\|_{\ell^2}^3\big),\end{aligned}$$

where we used Young's inequality and Cauchy-Schwarz inequality.

The inequality (5.28) is immediate since $\ell^2\subset\ell^\infty$.

To prove (5.29), we note

$$\Big\|k\sum_{k_1+k_2=k} e^{i3kk_1k_2s}v_{k_1}v_{k_2}\Big\|_{\ell^2_{|k|\le N}} \lesssim \|v*v\|_{\ell^\infty}\|k\|_{\ell^2_{|k|\le N}} \lesssim N^{3/2}\|v\|_{\ell^2}^2$$

It remains to prove (5.27). We estimate B_4 as

$$\begin{aligned}|B_4(v)_k| \lesssim &\sideset{}{^*}\sum_{k_1+k_2+k_3+k_4=k}\frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1+k_2||k_1+k_3+k_4||k_2+k_3+k_4|}\\ &+ \sideset{}{^*}\sum_{k_1+k_2+k_3+k_4=k}\frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1||k_1+k_2||k_2+k_3+k_4|}.\end{aligned}$$

We will estimate the first line, the same method works for the second one. By duality it suffices to estimate

$$\sup_{\|h\|_{\ell^2}=1} \sum_{k_1,k_2,k_3,k_4}^{*} \frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}||h_{k_1+k_2+k_3+k_4}|}{|k_1+k_2||k_1+k_3+k_4||k_2+k_3+k_4|}$$
$$\leq \sup_{\|h\|_{\ell^2=1}} \Big(\sum_{k_1,k_2,k_3,k_4}^{*} \frac{|v_{k_1}v_{k_4}|}{|k_1+k_2||k_1+k_3+k_4|} \Big)^{1/2} \Big(\sum_{k_1,k_2,k_3,k_4}^{*} \frac{|v_{k_2}v_{k_3}||h_{k_1+k_2+k_3+k_4}|}{|k_2+k_3+k_4|} \Big)^{1/2}$$
$$\lesssim \|v\|_{\ell^2}^4.$$

The estimate for the first sum follows by summing in the order k_2, k_3, k_1, k_4. For the second we sum in the order k_1, k_4, k_2, k_3. □

5.5. **Nonlinear Smoothing.** As we demonstrated, the restricted norm method is usually very efficient when one is treating dispersive PDE with nonlinearities that involve derivatives. For equations like the NLS where the nonlinearity is a simple monomial other methods work equally well. An example can be given by considering the classical Strichartz estimates. But for many applications it is useful to study the regularity properties of the equation in more details. One such instance is when one tries to prove that the nonlinear Duhamel term is in a smoother Sobolev space than the initial data (recall that the linear evolution is usually unitary on Sobolev norms so the full solution cannot lie in a smoother space). To prove such a statement one can implement again the $X^{s,b}$ method. In these notes we show how these estimates work in a simple example. In the next section we discuss the initial and boundary value problem (IBVP) for dispersive PDE. As an application we should mention that one can use these smoothing estimates to obtain persistence of regularity and uniqueness of solutions for rough initial data for IBVP on the half line. In the case of IBVP we will see that uniqueness is not straightforward.

To prove the first smoothing result we reproduce the argument in [27]. We start with the cubic NLS on the torus.

$$iu_t + u_{xx} + |u|^2 u = 0, \quad t \in \mathbb{R}, \ \ x \in \mathbb{T},$$

with initial data in H^s, $s > 0$. By similar methods we can of course establish local well–posedness in the corresponding $X^{s,b}$ space:

$$\|u\|_{X^{s,b}} = \big\| \widehat{u}(\tau,k) \langle k\rangle^s \langle \tau - k^2\rangle^b \big\|_{L^2_\tau \ell^2_k}.$$

Writing

$$\widehat{|u|^2 u}(k) = \sum_{k_1,k_2} \widehat{u}(k_1)\overline{\widehat{u}(k_2)}\widehat{u}(k-k_1+k_2)$$
$$= \frac{1}{\pi}\|u\|_2^2 \widehat{u}(k) - |\widehat{u}(k)|^2 \widehat{u}(k) + \sum_{k_1\neq k, k_2 \neq k_1} \widehat{u}(k_1)\overline{\widehat{u}(k_2)}\widehat{u}(k-k_1+k_2)$$
$$= \frac{1}{\pi}\|u(0)\|_2^2 \widehat{u}(k) - |\widehat{u}(k)|^2 \widehat{u}(k) + \widehat{R(u)}(k)$$
$$=: P\widehat{u}(k) + \widehat{\rho(u)}(k) + \widehat{R(u)}(k).$$

It is easy to see that the inverse Fourier transform of the first two summands are in $X^{s,b-1}$ if u is in $X^{s,b}$. The local well–posedness in $X^{s,b}$ for $s > 0$ and $b = \frac{1}{2}+$

follows from the following lemma in a standard way as above. We should note that the $X^{s,b}$ method is modified by introducing the factor P in the definition of the norm. This substitution does not alter what follows.

First recall the following notation (look at the exercises for details)

$$\phi_\beta(k) := \sum_{|n|\leq|k|} \frac{1}{|n|^\beta} \sim \begin{cases} 1, & \beta > 1, \\ \log(1+\langle k\rangle), & \beta = 1, \\ \langle k\rangle^{1-\beta}, & \beta < 1. \end{cases} \tag{5.30}$$

Proposition 5.28. *For fixed $s > 0$ and $a < \min(2s, 1/2)$, we have*

$$\|R(u)\|_{X^{s+a,b-1}} \lesssim \|u\|^3_{X^{s,b}}.$$

provided that $0 < b - 1/2$ is sufficiently small.

Proof.

$$\|R(u)\|^2_{X^{s+a,b-1}} = \Big\| \int_{\tau_1,\tau_2} \sum_{k_1\neq k, k_2\neq k_1} \frac{\langle k\rangle^{s+a}\widehat{u}(k_1,\tau_1)\overline{\widehat{u}(k_2,\tau_2)}\widehat{u}(k-k_1+k_2,\tau-\tau_1+\tau_2)}{\langle \tau - k^2\rangle^{1-b}} \Big\|^2_{\ell^2_k L^2_\tau}.$$

Let

$$f(k,\tau) = |\widehat{u}(k,\tau)|\langle k\rangle^s\langle \tau - k^2\rangle^b.$$

It suffices to prove that

$$\Big\| \int_{\tau_1,\tau_2} \sum_{k_1\neq k, k_2\neq k_1} M(k_1,k_2,k,\tau_1,\tau_2,\tau) f(k_1,\tau_1) f(k_2,\tau_2) f(k-k_1+k_2,\tau-\tau_1+\tau_2)\Big\|^2_{\ell^2_k L^2_\tau} \lesssim \|f\|_2^6,$$

where

$$M(k_1,k_2,k,\tau_1,\tau_2,\tau) = \frac{\langle k\rangle^{s+a}\langle k_1\rangle^{-s}\langle k_2\rangle^{-s}\langle k-k_1+k_2\rangle^{-s}}{\langle \tau-k^2\rangle^{1-b}\langle \tau_1-k_1^2\rangle^b\langle \tau_2-k_2^2\rangle^b\langle \tau-\tau_1+\tau_2-(k-k_1+k_2)^2\rangle^b}. \tag{5.31}$$

By the Cauchy–Schwarz inequality in τ_1,τ_2,k_1,k_2 variables, we estimate the norm above by

$$\sup_{k,\tau}\Big(\int_{\tau_1,\tau_2} \sum_{k_1\neq k, k_2\neq k_1} M^2(k_1,k_2,k,\tau_1,\tau_2,\tau)\Big)\times \Big\| \int_{\tau_1,\tau_2} \sum_{k_1,k_2} f^2(k_1,\tau_1) f^2(k_2,\tau_2) f^2(k-k_1+k_2,\tau-\tau_1+\tau_2)\Big\|_{\ell^1_k L^1_\tau}.$$

Note that the norm above is equal to $\|f^2 * f^2 * f^2\|_{\ell^1_k L^1_\tau}$, which can be estimated by $\|f\|_2^6$ by Young's inequality. Therefore, it suffices to prove that the supremum above is finite.

Using the exercises and integrating the τ_1 and τ_2 integrals, we obtain

$$\sup_{k,\tau}\int_{\tau_1,\tau_2}\sum_{k_1\neq k, k_2\neq k_1} M^2 \lesssim \sup_{k,\tau}\sum_{k_1\neq k, k_2\neq k_1} \frac{\langle k\rangle^{2s+2a}\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}\langle k-k_1+k_2\rangle^{-2s}}{\langle \tau-k^2\rangle^{2-2b}\langle \tau-k_1^2+k_2^2-(k-k_1+k_2)^2\rangle^{4b-1}}$$
$$\lesssim \sup_{k}\sum_{k_1\neq k, k_2\neq k_1} \frac{\langle k\rangle^{2s+2a}\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}\langle k-k_1+k_2\rangle^{-2s}}{\langle k^2-k_1^2+k_2^2-(k-k_1+k_2)^2\rangle^{2-2b}}.$$

The last line follows by the simple fact

$$\langle \tau - n\rangle\langle \tau - m\rangle \gtrsim \langle n - m\rangle. \tag{5.32}$$

Since we have only the nonresonant terms, it suffices to estimate

$$\sum_{k_1,k_2} \frac{\langle k\rangle^{2s+2a}\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}\langle k-k_1+k_2\rangle^{-2s}}{\langle k-k_1\rangle^{2-2b}\langle k_1-k_2\rangle^{2-2b}}.$$

To estimate this sum we consider the cases $|k-k_1+k_2| \gtrsim |k|$, $|k_1| \gtrsim |k|$, and $|k_2| \gtrsim |k|$.

In the first case, we bound the sum using the exercises (see also Lemma 6.8 in the next section) by

$$\sum_{|k_2-k_1|\ll|k|} \frac{\langle k\rangle^{2a}\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}}{\langle k-k_1\rangle^{2-2b}\langle k_1-k_2\rangle^{2-2b}} \lesssim \sum_{k_1} \frac{\langle k\rangle^{2a}\phi_{2s}(k_1)}{\langle k-k_1\rangle^{2-2b}\langle k_1\rangle^{2-2b+2s}} \lesssim 1,$$

provided $a < \min(2s, 1/2)$ and $0 < b - 1/2$ is sufficiently small.

The third case is similar.

In the second case, we similarly bound the sum by

$$\begin{aligned}
&\sum_{k_1,k_2} \frac{\langle k\rangle^{2a}}{\langle k-k_1\rangle^{2-2b}\langle k_1-k_2\rangle^{2-2b}\langle k-k_1+k_2\rangle^{2s}\langle k_2\rangle^{2s}}\\
&\quad\lesssim \sum_{k_1,k_2} \frac{\langle k\rangle^{2a}}{\langle k-k_2\rangle^{2-2b}\langle k_1-k_2\rangle^{2-2b}\langle k-k_1+k_2\rangle^{2s}\langle k_2\rangle^{2s}}\\
&\quad\quad+ \sum_{k_1,k_2} \frac{\langle k\rangle^{2a}}{\langle k-k_1\rangle^{2-2b}\langle k-k_2\rangle^{2-2b}\langle k-k_1+k_2\rangle^{2s}\langle k_2\rangle^{2s}}\\
&\quad\quad\quad\lesssim \langle k\rangle^{2a-4+4b}\phi_{2s}(k)^2 + \sum_{k_2} \frac{\langle k\rangle^{2a}\phi_{2s}(k_2)}{\langle k-k_2\rangle^{2-2b}\langle k_2\rangle^{2s+2-2b}} \lesssim 1.
\end{aligned}$$

□

This lemma also implies the following smoothing statement for NLS. We decompose the solution as

$$u(x,t) = e^{i(\partial_{xx}+P)t}g + \mathcal{N}(x,t),$$

where $P = \frac{\|g\|_2^2}{\pi}$. Here $\mathcal{N}$ is the nonresonant part of the nonlinear Duhamel term of the solution that is

$$\mathcal{N}(x,t) = \int_0^t e^{i(\partial_{xx}+P)(t-s)}\Big(\rho(u)(x,s) + R(u)(x,s)\Big)ds.$$

We first note that

$$\|\rho(u)\|_{H^{s+a}} = \sqrt{\sum_k |\widehat{u}(k)|^6\langle k\rangle^{2s+2a}} \lesssim \|u\|_{H^s}^3, \tag{5.33}$$

for $0 \le a \le 2s$. We thus have the following:

Proposition 5.29. *For fixed $s > 0$ and $a < \min(2s, 1/2)$, and for $|t| < \delta$ we have*

$$\|u(t) - e^{i(\partial_{xx} + \|u(0)\|_2^2/\pi)t} u(0)\|_{H^{s+a}} \lesssim \|u\|^3_{X^{s,b}_\delta}.$$

provided that $0 < b - 1/2$ is sufficiently small. Here $[-\delta, \delta]$ is the local existence interval.

6. Initial and Boundary value problems

We now turn our attention to dispersive PDE with non-homogeneous boundary conditions. We consider the case of the half–line, mainly for two reasons. Firstly because the theory is much harder for PDE posed on bounded intervals with general boundary data. And secondly because in the case of the semi–infinite line we can reformulate the problem appropriately and use the dispersive properties of the equations. More precisely we can use the powerful tools from Fourier Analysis that we have developed in previous sections. We should also note that the problem is much easier if one has zero Dirichlet boundary data but we cannot cover all the different cases in this short section. The easiest problem to consider is the cubic NLS. For the initial and boundary value problem (IBVP) for the KdV equation the reader can consult [13], [41], and [5].

We remark from the beginning that we cannot describe within the limitations of a short course all the methods that have been proposed in the past to resolve these problems even on the half–line. See for example [29] for the analysis of initial and boundary value problems that are based on complete integrability techniques. In this section we only present the aspects of the theory that are connected through the Fourier techniques that have been already presented in these notes.

We begin by studying the following initial-boundary value problem (IBVP)

$$\begin{aligned} &iu_t + u_{xx} + \lambda |u|^2 u = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\ &u(x,0) = g(x), \quad u(0,t) = h(t). \end{aligned} \tag{6.1}$$

Here $\lambda = \pm 1$, $g \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$. The compatibility condition is necessary since the solutions we are interested in are continuous space-time functions for $s > \frac{1}{2}$.

The term that models the nonlinear effects is cubic and the equation can be focusing ($\lambda = 1$) or defocusing ($\lambda = -1$). Nonlinear Schrödinger equations (NLS) of this form model a variety of physical phenomena in optics, water wave theory and Langmuir waves in a hot plasma, [70]. In the case of the semi infinite strip $(0, \infty) \times [0, T]$, the solution $u(x, t)$ of (6.1) models the amplitude of the wave generated at one end and propagating freely at the other. For an interesting example of such a wave train in deep water waves, see [1].

Our intention is to study this problem by using the tools that are available to us in the case of the full line. In this case the equation is strongly dispersive, and it has been studied extensively during the last 40 years. We use the restricted norm method (also known as the $X^{s,b}$ method) of Bourgain, [7, 8], modified appropriately. The idea to use the restricted norm method in the case of IBVP with mild nonlinearities comes from [13]. Their paper introduced a method to solve initial–boundary value problems for nonlinear dispersive partial differential equations by recasting these problems as initial value problems with an appropriate forcing term. This reformulation transports the robust theory of initial value problems to the initial–boundary value setting. The problem they considered was the Korteweg-de Vries equation on the half–line. In this case to recover the derivative in the nonlinearity one has to use the cancelations of the nonlinear waves that are nicely captured by the $X^{s,b}$ method. The idea of reformulating the problem as an initial value problem with forcing was applied in the case of the NLS with a general power nonlinearity in [39, 40]. The difference is that one has to use Strichartz estimates which are appropriate for dispersive equations with power type nonlinearities. For NLS on $\mathbb{R}^n$ the Strichartz estimates give sharp well–posedness results. One can also use more standard Laplace transform techniques to study (6.1), see e.g. [6]. This is based on an explicit solution formula of the linear nonhomogeneous boundary value problem

$$\begin{aligned} &iu_t + u_{xx} = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\ &u(x,0) = 0, \quad u(0,t) = h(t). \end{aligned} \tag{6.2}$$

which is obtain by formally using the Laplace transform. One then can use Duhamel's formula and express the nonlinear solution as a superposition of the linear evolution which incorporates the boundary term and the initial data with the nonlinearity.

In this note we thus combine the Laplace transform method [6] with the $X^{s,b}$ method [7] to prove that the nonlinear part of the solution is smoother than the initial data. More precisely, we prove

Theorem 6.1. *Fix $s \in (0, \frac{5}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$, $g \in H^s(\mathbb{R}^+)$, and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$. Then, for t in the local existence interval $[0, T]$ and $a < \min(2s, \frac{1}{2}, \frac{5}{2} - s)$ we have*

$$u(x,t) - W_0^t(g,h) \in C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^+),$$

where $W_0^t(g,h)$ is the solution of the corresponding linear equation (6.1) with $\lambda = 0$.

We should note that the nonlinear estimates that are required to prove the above theorem also prove that the IBVP is locally well–posed in H^s for $s \in (0, \frac{5}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$. In particular we have the following Theorem:

Theorem 6.2. *Fix $s \in (0, \frac{5}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$. Then (6.1) is locally wellposed in $H^s(\mathbb{R}^+)$.*

As we have discussed above, it is usual practice in the theory of nonlinear PDE to first try and find the right Banach spaces that the solutions live in, and then prove nonlinear estimates in these spaces. This process at the end delivers the solution of the IBVP as a fixed point of a nonlinear solution map (Duhamel's formula). The selection of the spaces is dictated by the linear estimates since we recast the equations as a perturbation of the linear evolution. Thus for $g \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$, we are looking for a solution

$$u \in X^{s,b}(\mathbb{R} \times [0,T]) \cap C_t^0 H_x^s([0,T] \times \mathbb{R}) \cap C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R} \times [0,T]). \tag{6.3}$$

It is a well known fact that (see (6.12) below for the definition of the $X^{s,b}$ norm)

$$u \in X^{s,b}(\mathbb{R} \times [0,T]) \subset C_t^0 H_x^s([0,T] \times \mathbb{R})$$

for any $b > \frac{1}{2}$. However, to close the fixed point argument we need to take $b < \frac{1}{2}$. For this reason we need to prove the continuity of the solution directly via additional estimates for the linear evolution $W_0^t(g,h)$ (corresponding to (6.1) with $\lambda = 0$). The reader should keep in mind that we estimate two distinct linear processes. One is the usual solution of the free Schrödinger equation with initial data g which we denote by $W_{\mathbb{R}}g$ and the other is the linear solution, $W_0^t(0,h)$ to the IBVP (6.2).

We define $H^s(\mathbb{R}^+)$ norm as

$$\|g\|_{H^s(\mathbb{R}^+)} := \inf\left\{\|\tilde{g}\|_{H^s(\mathbb{R})} : \tilde{g}(x) = g(x),\, x > 0\right\}.$$

Note that we have $\|g'\|_{H^{s-1}(\mathbb{R}^+)} \leq \|g\|_{H^s(\mathbb{R}^+)}$. If $g \in H^s(\mathbb{R}^+)$ for some $s > \frac{1}{2}$, take an extension $\tilde{g} \in H^s(\mathbb{R})$. By Sobolev embedding $\tilde{g}$ is continuous on $\mathbb{R}$, and hence $g(0)$ is well defined. We have the following lemma concerning extensions of $H^s(\mathbb{R}^+)$ functions.

Lemma 6.3. *Let $h \in H^s(\mathbb{R}^+)$ for some $-\frac{1}{2} < s < \frac{5}{2}$.*
i) If $-\frac{1}{2} < s < \frac{1}{2}$, then $\|\chi_{(0,\infty)}h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.
ii) If $\frac{1}{2} < s < \frac{3}{2}$ and $h(0) = 0$, then $\|\chi_{(0,\infty)}h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.

The proof of this Lemma is essentially in [13]. We can summarize here a different proof by observing that the first part follows from the weighted L^2 boundedness of Hilbert transform and the fact that $\langle\xi\rangle^{2s}$ is an A_2 weight for $s \in (-\frac{1}{2}, \frac{1}{2})$. As for the second we note that, since $h(0) = 0$, the distributional derivative of $\chi_{(0,\infty)}h$ is $\chi_{(0,\infty)}h'$, and then we can use i).

To construct the solutions of (6.1) we first consider the linear problem:

$$\begin{aligned} &iu_t + u_{xx} = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\ &u(x,0) = g(x) \in H^s(\mathbb{R}^+), \quad u(0,t) = h(t) \in H^{\frac{2s+1}{4}}(\mathbb{R}^+), \end{aligned} \tag{6.4}$$

with the compatibility condition $h(0) = g(0)$ for $s > \frac{1}{2}$. Note that the uniqueness of the solutions of equation (6.4) follows by considering the equation with $g = h = 0$ with the method of odd extension. We now construct the unique solution of (6.4), that we denote by $W_0^t(g,h)$, for $t \in [0,1]$. Note that

$$W_0^t(g,h) = W_0^t(0, h-p) + W_{\mathbb{R}}(t)g_e,$$

where g_e is an H^s extension of g to $\mathbb{R}$ satisfying $\|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}$. Moreover,

$$p(t) = \eta(t)[W_{\mathbb{R}}(t)g_e]\big|_{x=0},$$

which is well-defined and is in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ by Lemma 6.5 below and $\eta(t)$ is a bump function. The properties of the free Schrödinger evolution are well known. To understand the first summand, $W_0^t(0,h)$, consider the linear boundary value problem (6.2) with $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$.

To find analytically the solution $W_0^t(0,h)$, recall that for a suitable function $f(t)$ the Laplace transform is defined for $\Re s > 0$ as

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt$$

and we have the inversion formula

$$f(t) = \lim_{\gamma\to\infty} \left(\frac{1}{2\pi i} \int_{\sigma - i\gamma}^{\sigma + i\gamma} \mathcal{L}(f)(s) e^{st} ds \right)$$

for $t > 0$ and $|f(t)| \leq e^{Mt}$ for some positive real number M and $\sigma \in \mathbb{R}$ such that $\sigma > M$. If we take the Laplace transform with respect to t in equation (6.4) we convert the linear initial and boundary value problem to the following one parameter second order boundary value problem

$$\begin{aligned} & is\mathcal{L}u(x,s) + (\mathcal{L}u(x,s))_{xx} = 0, \\ & \mathcal{L}u(0,s) = \mathcal{L}(h)(s), \quad \mathcal{L}u(+\infty, s) = 0 \end{aligned} \tag{6.5}$$

where $\mathcal{L}(u)(x,s)$ is the Laplace transform of $u(x,t)$ and $\Re s > 0$. To solve this problem we try the the solution

$$\mathcal{L}(u)(x,s) = e^{\alpha(s)x} \mathcal{L}(h)(s)$$

and obtain the algebraic equation

$$is + \alpha^2 = 0$$

for $\Re\alpha < 0$. If we invert, we formally obtain

$$u(x,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{L}(h)(s) e^{st} e^{\alpha(s)x} ds$$

for $x,\ t > 0$ and $\sigma > 0$ fixed. If we let $\sigma \to 0$ (and substitute $s = i\beta$) we obtain with $-\beta + \alpha^2 = 0$, and $\Re\alpha \leq 0$,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}(h)(i\beta) e^{i\beta t} e^{\alpha(i\beta)x} d\beta.$$

For $\beta \geq 0$ we solve for $\alpha = -\sqrt{\beta}$ and for $\beta < 0$ we solve for $\alpha = i\sqrt{-\beta}$. Then

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^0 \mathcal{L}(h)(i\beta) e^{i\beta t} e^{i\sqrt{-\beta}x} d\beta + \frac{1}{2\pi} \int_0^\infty \mathcal{L}(h)(i\beta) e^{i\beta t} e^{-\sqrt{\beta}x} d\beta =$$

$$\frac{1}{2\pi} \int_0^\infty \mathcal{L}(h)(-i\beta) e^{-i\beta t} e^{i\sqrt{\beta}x} d\beta + \frac{1}{2\pi} \int_0^\infty \mathcal{L}(h)(i\beta) e^{i\beta t} e^{-\sqrt{\beta}x} d\beta$$

and by the substitution $\beta \to \beta^2$ we obtain the representation

$$u(x,t) = \frac{1}{\pi} \int_0^\infty \beta\mathcal{L}(h)(-i\beta^2) e^{-i\beta^2 t} e^{i\beta x} d\beta + \frac{1}{\pi} \int_0^\infty \beta\mathcal{L}(h)(i\beta^2) e^{i\beta^2 t} e^{-\beta x} d\beta.$$

It is now clear that we can write the solution as $W_0^t(0,h) = W_1 h + W_2 h$, where

$$W_1 h(x,t) = \frac{1}{\pi} \int_0^\infty e^{-i\beta^2 t + i\beta x} \beta \widehat{h}(-\beta^2) d\beta, \tag{6.6}$$

$$W_2 h(x,t) = \frac{1}{\pi} \int_0^\infty e^{i\beta^2 t - \beta x} \beta \widehat{h}(\beta^2) d\beta. \tag{6.7}$$

Here by a slight abuse of notation

$$\widehat{h}(\xi) = \mathcal{F}\big(\chi_{(0,\infty)}h\big)(\xi) = \int_0^\infty e^{-i\xi t} h(t)dt. \tag{6.8}$$

By a change of variable and Lemma 6.3, under the conditions above we have

$$\sqrt{\int_0^\infty \langle\beta\rangle^{2s}\big|\beta\widehat{h}(\pm\beta^2)\big|^2 d\beta} \lesssim \|\chi_{(0,\infty)}h\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim \|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}. \tag{6.9}$$

This simple calculation is used repeatedly in what follows.

Note that W_1 is well-defined for $x, t \in \mathbb{R}$. We also extend W_2 to all x by

$$W_2h(x,t) = \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t - \beta x}\rho(\beta x)\beta\widehat{h}(\beta^2)d\beta, \tag{6.10}$$

where $\rho(x)$ is a smooth function supported on $(-2,\infty)$, and $\rho(x) = 1$ for $x > 0$.

Therefore the solution of (6.4) for $t \in [0,1]$ is given by

$$W_0^t(g,h) = W_0^t(0, h-p) + W_{\mathbb{R}}(t)g_e, \quad p(t) = \eta(t)[W_{\mathbb{R}}(t)g_e](0).$$

We note that $W_0^t(g,h)$ is well-defined for $x, t \in \mathbb{R}$, and its restriction to $\mathbb{R}^+ \times [0,1]$ is independent of the extension g_e.

Consider now the integral equation

$$u(t) = \eta(t)W_{\mathbb{R}}(t)g_e + \eta(t)\int_0^t W_{\mathbb{R}}(t-t')F(u)\,dt' + \eta(t)W_0^t\big(0, h-p-q\big)(t), \tag{6.11}$$

where

$$F(u) = \eta(t/T)|u|^2u, \quad p(t) = \eta(t)D_0(W_{\mathbb{R}}g_e), \quad \text{and}$$

$$q(t) = \eta(t)D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')F(u)\,dt'\Big).$$

Here $D_0f(t) = f(0,t)$, and g_e is an H^s extension of g to $\mathbb{R}$. In what follows we will prove that the integral equation (6.11) has a unique solution in a suitable Banach space on $\mathbb{R}\times\mathbb{R}$ for some $T < 1$. Using the definition of the boundary operator, it is clear that the restriction of u to $\mathbb{R}^+ \times [0,T]$ satisfies (6.1) in the distributional sense. Also note that the smooth solutions of (6.11) satisfy (6.1) in the classical sense.

We work with the space $X^{s,b}(\mathbb{R}\times\mathbb{R})$ [7, 8]:

$$\|u\|_{X^{s,b}} = \big\|\widehat{u}(\tau,\xi)\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b\big\|_{L^2_\tau L^2_\xi}. \tag{6.12}$$

For completeness we summarize one more time the basic properties of the $X^{s,b}$ norms.

First recall the embedding $X^{s,b} \subset C^0_t H^s_x$ for $b > \frac{1}{2}$ and the following inequalities from [7, 30]. First the fact that for any s, b we have

$$\|\eta(t)W_{\mathbb{R}}g\|_{X^{s,b}} \lesssim \|g\|_{H^s}. \tag{6.13}$$

In addition we have for any $s \in \mathbb{R}$, $0 \le b_1 < \frac{1}{2}$, and $0 \le b_2 \le 1 - b_1$

$$\Big\|\eta(t)\int_0^t W_{\mathbb{R}}(t-t')F(t')dt'\Big\|_{X^{s,b_2}} \lesssim \|F\|_{X^{s,-b_1}}. \tag{6.14}$$

Finally, for $T < 1$, and $-\frac{1}{2} < b_1 < b_2 < \frac{1}{2}$, we have

$$\|\eta(t/T)F\|_{X^{s,b_1}} \lesssim T^{b_2-b_1}\|F\|_{X^{s,b_2}}. \tag{6.15}$$

Our solutions are characterized by the following definition:

Definition 6.4. *We say (6.1) is locally well–posed in $H^s(\mathbb{R}^+)$, if for any $g \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$, the equation (6.11) has a unique solution in*

$$X^{s,b}(\mathbb{R}\times[0,T]) \cap C_t^0H_x^s([0,T]\times\mathbb{R}) \cap C_x^0H_t^{\frac{2s+1}{4}}(\mathbb{R}\times[0,T]),$$

for any $b < \frac{1}{2}$. Moreover, if u and v are two such solutions coming from different extensions g_{e1}, g_{e2}, then their restriction to $[0,\infty)\times[0,T]$ are the same. Furthermore, if $g_n \to g$ in $H^s(\mathbb{R}^+)$ and $h_n \to h$ in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, then $u_n \to u$ in the space above.

6.1. **Estimates for linear terms.** We start with the following well known Kato smoothing estimate converting space derivatives to time derivatives. This estimate justifies the choice of spaces concerning g, h in (6.1).

Lemma 6.5. *(Kato smoothing inequality) Fix $s \ge 0$. For any $g \in H^s(\mathbb{R})$, we have $\eta(t)W_{\mathbb{R}}g \in C_x^0H_t^{\frac{2s+1}{4}}(\mathbb{R}\times\mathbb{R})$, and we have*

$$\big\|\eta W_{\mathbb{R}}g\big\|_{L_x^\infty H_t^{\frac{2s+1}{4}}} \lesssim \|g\|_{H^s(\mathbb{R})}.$$

Proof. Exercise.

□

Lemma 6.6 and Proposition 6.7 below show that the boundary operator belongs to the space (6.3).

Lemma 6.6. *Let $s \ge 0$. Then for h satisfying $\chi_{(0,\infty)}h \in H^{\frac{2s+1}{4}}(\mathbb{R})$, we have $W_0^t(0,h) \in C_t^0H_x^s(\mathbb{R}\times\mathbb{R})$, and $\eta(t)W_0^t(0,h) \in C_x^0H_t^{\frac{2s+1}{4}}(\mathbb{R}\times\mathbb{R})$.*

Proof. We start with the claim $W_2h \in C_t^0H_x^s(\mathbb{R}\times\mathbb{R})$. Let $f(x) = e^{-x}\rho(x)$. Note that f is a Schwartz function. Recalling (6.10), we have

$$W_2h = \int_0^\infty f(\beta x)e^{i\beta^2t}\beta\widehat{h}(\beta^2)d\beta = \int_{\mathbb{R}} f(\beta x)\mathcal{F}\big(e^{-it\Delta}\psi\big)(\beta)d\beta,$$

where

$$\widehat{\psi}(\beta) = \beta\widehat{h}(\beta^2)\chi_{[0,\infty)}(\beta).$$

Note that by (6.8) and (6.9), $\|\psi\|_{H^s} \lesssim \|\chi_{(0,\infty)}h\|_{H^{\frac{2s+1}{4}}(\mathbb{R})}$. Using this and the continuity of $e^{-it\Delta}$ in H^s, it suffices to prove that

$$Tg(x) := \int_{\mathbb{R}} f(\beta x)\widehat{g}(\beta)d\beta$$

is bounded in H^s for $s \geq 0$. This follows from the case $s = 0$ noting that

$$\partial_x^s Tg(x) = \int_{\mathbb{R}} f^{(s)}(\beta x)\beta^s \widehat{g}(\beta)d\beta, \quad s \in \mathbb{N},$$

and by interpolation. For $s = 0$, after the change of variable $\beta x \to \beta$, we have

$$Tg(x) = \int_{\mathbb{R}} f(\beta)x^{-1}\widehat{g}(\beta x^{-1})d\beta.$$

Therefore,

$$\|Tg\|_{L^2} \leq \int_{\mathbb{R}} |f(\beta)| \big\|x^{-1}\widehat{g}(\beta x^{-1})\big\|_{L^2_x} d\beta.$$

Noting that

$$\big\|x^{-1}\widehat{g}(\beta x^{-1})\big\|_{L^2_x}^2 = \int_{\mathbb{R}} x^{-2}|\widehat{g}(\beta x^{-1})|^2 dx = \int_{\mathbb{R}} \beta^{-1}|\widehat{g}(y)|^2 dy = \beta^{-1}\|g\|_{L^2}^2,$$

we obtain

$$\|Tg\|_{L^2} \leq \|g\|_{L^2} \int_{\mathbb{R}} |f(\beta)| \frac{d\beta}{\sqrt{\beta}} \lesssim \|g\|_{L^2},$$

since $f \in \mathcal{S}$. This proves that $W_2 h \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$.

To prove that $\eta(t)W_2 h \in C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R} \times \mathbb{R})$, write

$$W_2 h = \int_{\mathbb{R}} f(\beta x)\mathcal{F}\big(e^{-it\Delta}\psi\big)(\beta)d\beta = \int_{\mathbb{R}} \frac{1}{x}\widehat{f}(\xi/x)(e^{-it\Delta}\psi)(\xi)d\xi = \int_{\mathbb{R}} \widehat{f}(\xi)(e^{-it\Delta}\psi)(x\xi)d\xi.$$

The claim follows from the using Kato smoothing and dominated convergence theorem noting that $\widehat{f} \in L^1$.

Finally, note that

$$W_1 h = W_{\mathbb{R}}\psi, \tag{6.16}$$

where

$$\widehat{\psi}(\beta) = \beta\widehat{h}(-\beta^2)\chi_{[0,\infty)}(\beta).$$

The claim follows as above from (6.8), (6.9), the continuity of $W_{\mathbb{R}}(t)$, and Kato smoothing Lemma 6.5. □

Proposition 6.7. *Let $b \leq \frac{1}{2}$ and $s \geq 0$. Then for h satisfying $\chi_{(0,\infty)}h \in H^{\frac{2s+1}{4}}(\mathbb{R})$, we have*

$$\|\eta(t)W_0^t(0,h)\|_{X^{s,b}} \lesssim \|\chi_{(0,\infty)}h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}.$$

Proof. As before, define ψ as

$$\widehat{\psi}(\beta) = \beta\widehat{h}(-\beta^2)\chi_{(0,\infty)}(\beta).$$

Using (6.16), (6.13), (6.8), and (6.9), we have

$$\|\eta W_1 h\|_{X^{s,b}} = \|\eta W_{\mathbb{R}}(t)\psi\|_{X^{s,b}} \lesssim \|\psi\|_{H^s} \lesssim \|\chi_{(0,\infty)}h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}.$$

For W_2, by interpolation, it suffices to prove the statement for $s = 0, 1, 2, ...$ Let $f(x) = e^{-x}\rho(x)$. Note that

$$\partial_x^s \eta W_2 h = \eta \int_0^\infty e^{i\beta^2 t} f^{(s)}(\beta x)\beta^{s+1}\widehat{h}(\beta^2)d\beta.$$

Therefore, it suffices to prove the inequality for $s = 0$ and $b = \frac{1}{2}$. We have

$$\widehat{\eta W_2 h}(\xi, \tau) = \int_0^\infty \widehat{\eta}(\tau - \beta^2)\widehat{f}(\xi/\beta)\widehat{h}(\beta^2)d\beta.$$

Since f is a Schwartz function, we have

$$\big|\widehat{f}(\xi/\beta)\big| \lesssim \frac{1}{1 + \xi^2/\beta^2} = \frac{\beta^2}{\beta^2 + \xi^2}.$$

Therefore

$$\|\eta W_2 h\|_{X^{0,\frac{1}{2}}} \lesssim \Big\|\langle\tau + \xi^2\rangle^{\frac{1}{2}} \int_0^\infty |\widehat{\eta}(\tau - \beta^2)| \frac{\beta^2}{\beta^2 + \xi^2}|\widehat{h}(\beta^2)|d\beta\Big\|_{L^2_\xi L^2_\tau}.$$

We divide this integral into pieces $\xi^2 + \beta^2 > 1$ and $\xi^2 + \beta^2 \leq 1$. In the former case using $|\widehat{\eta}(\tau - \beta^2)| \lesssim \langle\tau - \beta^2\rangle^{-3}$, $\langle\tau + \xi^2\rangle \lesssim \langle\tau - \beta^2\rangle\langle\beta^2 + \xi^2\rangle$, and $\beta^2 + \xi^2 \sim \langle\beta^2 + \xi^2\rangle$, we have the bound

$$\Big\|\int_0^\infty \langle\tau - \beta^2\rangle^{-2} \frac{\beta^2}{(\beta^2 + \xi^2)^{\frac{1}{2}}}|\widehat{h}(\beta^2)|d\beta\Big\|_{L^2_\xi L^2_\tau}.$$

Using Minkowski's and Young's inequalities, we have

$$\begin{aligned}
&\lesssim \Big\|\int_0^\infty \langle\tau - \beta^2\rangle^{-2} \Big\|\frac{\beta^2}{(\beta^2 + \xi^2)^{\frac{1}{2}}}\Big\|_{L^2_\xi}|\widehat{h}(\beta^2)|d\beta\Big\|_{L^2_\tau} \lesssim \Big\|\int_0^\infty \langle\tau - \beta^2\rangle^{-2}\beta^{\frac{3}{2}}|\widehat{h}(\beta^2)|d\beta\Big\|_{L^2_\tau} \\
&\lesssim \Big\|\int_0^\infty \langle\tau - \rho\rangle^{-2}\rho^{\frac{1}{4}}|\widehat{h}(\rho)|d\rho\Big\|_{L^2_\tau} \lesssim \|\langle\cdot\rangle^{-2}\|_{L^1}\big\|\rho^{\frac{1}{4}}\widehat{h}(\rho)\big\|_{L^2_\rho} \lesssim \|\chi_{(0,\infty)}h\|_{H^{\frac{1}{4}}(\mathbb{R})}.
\end{aligned}$$

In the latter case, we have the bound

$$\Big\|\langle\tau\rangle^{\frac{1}{2}} \int_0^1 \langle\tau\rangle^{-3} \frac{\beta^2}{\beta^2 + \xi^2}|\widehat{h}(\beta^2)|d\beta\Big\|_{L^2_{|\xi|\leq 1} L^2_\tau}.$$

Using Minkowski's inequality for both L^2 norms we have

$$\begin{aligned}
&\lesssim \int_0^1 \Big\|\frac{\beta^2}{\beta^2 + \xi^2}\Big\|_{L^2_{|\xi|\leq 1}}|\widehat{h}(\beta^2)|d\beta \lesssim \int_0^1 \beta^{\frac{1}{2}}|\widehat{h}(\beta^2)|d\beta \\
&\qquad\qquad \lesssim \int_0^1 \rho^{-\frac{1}{4}}|\widehat{h}(\rho)|d\rho \lesssim \|\chi_{(0,\infty)}h\|_{L^2(\mathbb{R})} \leq \|\chi_{(0,\infty)}h\|_{H^{\frac{1}{4}}(\mathbb{R})}.
\end{aligned}$$

In the second to last bound we used the Cauchy-Schwarz inequality. □

6.2. **Estimates for the nonlinear term.** In this subsection we establish estimates for the nonlinear term in (6.11) in order to close the fixed point argument and to obtain the smoothing theorem. Before we prove the main propositions of this section we need two Lemmas. The first one was proved in the exercises.

Lemma 6.8. *If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$, then*

$$\int \frac{1}{\langle x - a_1\rangle^{\beta}\langle x - a_2\rangle^{\gamma}} dx \lesssim \langle a_1 - a_2\rangle^{-\gamma}\phi_{\beta}(a_1 - a_2),$$

where

$$\phi_{\beta}(a) \sim \begin{cases} 1 & \beta > 1 \\ \log(1 + \langle a\rangle) & \beta = 1 \\ \langle a\rangle^{1-\beta} & \beta < 1. \end{cases}$$

Lemma 6.9. *For fixed $\rho \in (\frac{1}{2}, 1)$, we have*

$$\int \frac{1}{\langle x\rangle^{\rho}\sqrt{|x - a|}} dx \lesssim \frac{1}{\langle a\rangle^{\rho - \frac{1}{2}}}.$$

Proof. Let $A = \{x : |x - a| > 1\}$, and $B = \{x : |x - a| \leq 1\}$. Note that

$$\int_B \frac{1}{\langle x\rangle^{\rho}\sqrt{|x - a|}} dx \lesssim \frac{1}{\langle a\rangle^{\rho}} \int_B \frac{1}{\sqrt{|x - a|}} dx \lesssim \frac{1}{\langle a\rangle^{\rho}}.$$

Finally, using Lemma 6.8, we have

$$\int_A \frac{1}{\langle x\rangle^{\rho}\sqrt{|x - a|}} dx \lesssim \int_A \frac{1}{\langle x\rangle^{\rho}\sqrt{\langle x - a\rangle}} dx \lesssim \frac{1}{\langle a\rangle^{\rho - \frac{1}{2}}}.$$

□

Proposition 6.10. *For any smooth compactly supported function η, we have*

$$\Big\|\eta \int_0^t W_{\mathbb{R}}(t - t')F dt'\Big\|_{C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R}\times\mathbb{R})} \lesssim \begin{cases} \|F\|_{X^{s,-b}} & \text{for } 0 \leq s \leq \frac{1}{2}, b < \frac{1}{2}, \\ \|F\|_{X^{\frac{1}{2}, \frac{2s-1-4b}{4}}} + \|F\|_{X^{s,-b}} & \text{for } \frac{1}{2} \leq s \leq \frac{5}{2}, b < \frac{1}{2}. \end{cases}$$

Proof. The proof is based on an argument from [13].

It suffices to prove the bound above for $\eta D_0\big(\int_0^t W_{\mathbb{R}}(t-t')F dt'\big)$ since $X^{s,b}$ norm is independent of space translation. The continuity in x follows from this by dominated convergence theorem as in the proof of Lemma 6.5. First we consider the case $0 \leq s \leq \frac{1}{2}$. Note that

$$D_0\Big(\int_0^t W_{\mathbb{R}}(t - t')F dt'\Big) = \int_{\mathbb{R}} \int_0^t e^{-i(t-t')\xi^2} F(\widehat{\xi}, t') dt' d\xi.$$

Using

$$F(\widehat{\xi}, t') = \int_{\mathbb{R}} e^{it'\lambda} \widehat{F}(\xi, \lambda) d\lambda,$$

and

$$\int_0^t e^{it'(\xi^2+\lambda)} dt' = \frac{e^{it(\xi^2+\lambda)} - 1}{i(\lambda + \xi^2)}$$

we obtain

$$D_0\Big(\int_0^t W_{\mathbb{R}}(t - t')F dt'\Big) = \int_{\mathbb{R}^2} \frac{e^{it\lambda} - e^{-it\xi^2}}{i(\lambda + \xi^2)} \widehat{F}(\xi, \lambda) d\xi d\lambda.$$

Let ψ be a smooth cutoff for $[-1,1]$, and let $\psi^c = 1-\psi$. We write

$$\eta(t)D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')F dt'\Big) = \eta(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}-e^{-it\xi^2}}{i(\lambda+\xi^2)}\psi(\lambda+\xi^2)\widehat{F}(\xi,\lambda)d\xi d\lambda$$

$$+\eta(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}}{i(\lambda+\xi^2)}\psi^c(\lambda+\xi^2)\widehat{F}(\xi,\lambda)d\xi d\lambda - \eta(t)\int_{\mathbb{R}^2}\frac{e^{-it\xi^2}}{i(\lambda+\xi^2)}\psi^c(\lambda+\xi^2)\widehat{F}(\xi,\lambda)d\xi d\lambda$$
$$=: I + II + III.$$

By Taylor expansion, we have

$$\frac{e^{it\lambda}-e^{-it\xi^2}}{i(\lambda+\xi^2)} = ie^{it\lambda}\sum_{k=1}^{\infty}\frac{(-it)^k}{k!}(\lambda+\xi^2)^{k-1}$$

Therefore, we have

$$\|I\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim \sum_{k=1}^{\infty}\frac{\|\eta(t)t^k\|_{H^1}}{k!}\Big\|\int_{\mathbb{R}^2}e^{it\lambda}(\lambda+\xi^2)^{k-1}\psi(\lambda+\xi^2)\widehat{F}(\xi,\lambda)d\xi d\lambda\Big\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}$$
$$\lesssim \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\Big\|\langle\lambda\rangle^{\frac{2s+1}{4}}\int_{\mathbb{R}}(\lambda+\xi^2)^{k-1}\psi(\lambda+\xi^2)\widehat{F}(\xi,\lambda)d\xi\Big\|_{L^2_\lambda}$$
$$\lesssim \Big\|\langle\lambda\rangle^{\frac{2s+1}{4}}\int_{\mathbb{R}}\psi(\lambda+\xi^2)|\widehat{F}(\xi,\lambda)|d\xi\Big\|_{L^2_\lambda}.$$

By Cauchy-Schwarz inequality in ξ, we estimate this by

$$\Big[\int_{\mathbb{R}}\langle\lambda\rangle^{\frac{2s+1}{2}}\Big(\int_{|\lambda+\xi^2|<1}\langle\xi\rangle^{-2s}d\xi\Big)\Big(\int_{|\lambda+\xi^2|<1}\langle\xi\rangle^{2s}|\widehat{F}(\xi,\lambda)|^2 d\xi\Big)d\lambda\Big]^{1/2}$$
$$\lesssim \|F\|_{X^{s,-b}}\sup_{\lambda}\Big(\langle\lambda\rangle^{\frac{2s+1}{2}}\int_{|\lambda+\xi^2|<1}\langle\xi\rangle^{-2s}d\xi\Big)^{1/2} \lesssim \|F\|_{X^{s,-b}}.$$

The last inequality follows by a calculation substituting $\rho=\xi^2$.

For the second term, we have

$$\|II\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim \|\eta\|_{H^1}\Big\|\langle\lambda\rangle^{\frac{2s+1}{4}}\int_{\mathbb{R}}\frac{1}{\lambda+\xi^2}\psi^c(\lambda+\xi^2)\widehat{F}(\xi,\lambda)d\xi\Big\|_{L^2_\lambda}$$
$$\lesssim \Big\|\langle\lambda\rangle^{\frac{2s+1}{4}}\int_{\mathbb{R}}\frac{1}{\langle\lambda+\xi^2\rangle}|\widehat{F}(\xi,\lambda)|d\xi\Big\|_{L^2_\lambda}.$$

By Cauchy-Schwarz inequality in ξ, we estimate this by

$$\Big[\int_{\mathbb{R}}\langle\lambda\rangle^{\frac{2s+1}{2}}\Big(\int\frac{1}{\langle\lambda+\xi^2\rangle^{2-2b}\langle\xi\rangle^{2s}}d\xi\Big)\Big(\int\frac{\langle\xi\rangle^{2s}}{\langle\lambda+\xi^2\rangle^{2b}}|\widehat{F}(\xi,\lambda)|^2 d\xi\Big)d\lambda\Big]^{1/2}$$
$$\lesssim \|F\|_{X^{s,-b}}\sup_{\lambda}\Big(\langle\lambda\rangle^{\frac{2s+1}{2}}\int\frac{1}{\langle\lambda+\xi^2\rangle^{2-2b}\langle\xi\rangle^{2s}}d\xi\Big)^{1/2} \lesssim \|F\|_{X^{s,-b}}.$$

To obtain the last inequality recall that $s\le\frac12, b<\frac12$, and consider the cases $|\xi|<1$ and $|\xi|\ge 1$ separately. In the former case use $\langle\lambda+\xi^2\rangle\sim\langle\lambda\rangle$, and in the latter case use Lemma 6.8 after the change of variable $\rho=\xi^2$.

To estimate $\|III\|_{H^{\frac{2s+1}{4}}(\mathbb{R})}$, we divide the ξ integral into two pieces, $|\xi|\ge 1$, $|\xi|<1$. We estimate the contribution of the former piece as above (after the change of variable $\rho=\xi^2$):

$$\Big\|\langle\rho\rangle^{\frac{2s+1}{4}}\int_{\mathbb{R}}\frac{1}{\lambda+\rho}\psi^c(\lambda+\rho)\widehat{F}(\sqrt{\rho},\lambda)\frac{d\lambda}{\sqrt{\rho}}\Big\|_{L^2_{|\rho|\geq 1}}\lesssim\Big\|\langle\rho\rangle^{\frac{2s-1}{4}}\int_{\mathbb{R}}\frac{1}{\langle\lambda+\rho\rangle}|\widehat{F}(\sqrt{\rho},\lambda)|\,d\lambda\Big\|_{L^2_{|\rho|\geq 1}}.$$

By Cauchy-Schwarz in λ integral, and using $b<\frac{1}{2}$, we bound this by

$$\Big[\int_{|\rho|>1}\int_{\mathbb{R}}\frac{\langle\rho\rangle^{\frac{2s-1}{2}}}{\langle\lambda+\rho\rangle^{2b}}|\widehat{F}(\sqrt{\rho},\lambda)|^2d\lambda d\rho\Big]^{1/2}\lesssim\|F\|_{X^{s,-b}}.$$

We estimate the contribution of the latter term by

$$\int_{\mathbb{R}^2}\frac{\|\eta(t)e^{-it\xi^2}\|_{H^{\frac{2s+1}{4}}}\chi_{[-1,1]}(\xi)}{|\lambda+\xi^2|}\psi^c(\lambda+\xi^2)|\widehat{F}(\xi,\lambda)|d\xi d\lambda\lesssim\int_{\mathbb{R}^2}\frac{\chi_{[-1,1]}(\xi)}{\langle\lambda+\xi^2\rangle}|\widehat{F}(\xi,\lambda)|d\xi d\lambda.$$

For $b<\frac{1}{2}$, this is bounded by $\|F\|_{X^{0,-b}}$ by Cauchy-Schwarz inequality in ξ and λ integrals.

This finishes the proof for $0\leq s\leq\frac{1}{2}$.

For $s=\frac{5}{2}$, $\frac{2s+1}{4}=\frac{3}{2}$, we use the inequality

$$\|f\|_{H^{\frac{3}{2}}}\lesssim\|f\|_{L^2}+\|f'\|_{\dot{H}^{\frac{1}{2}}}.$$

The required bound for the L^2 norm follows from the $H^{\frac{1}{2}}$ bound above.

Note that

$$\frac{d}{dt}\Big[\eta(t)D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')Fdt'\Big)\Big]$$
$$=\eta'(t)D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')Fdt'\Big)+i\eta(t)\int_{\mathbb{R}^2}\frac{\lambda e^{it\lambda}+\xi^2e^{-it\xi^2}}{\lambda+\xi^2}\widehat{F}(\xi,\lambda)d\xi d\lambda$$
$$=\eta'(t)D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')Fdt'\Big)$$
$$+i\eta(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}-e^{-it\xi^2}}{\lambda+\xi^2}(-\xi^2)\widehat{F}(\xi,\lambda)d\xi d\lambda+i\eta(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}}{\langle\lambda+\xi^2\rangle}\langle\lambda+\xi^2\rangle\widehat{F}(\xi,\lambda)d\xi d\lambda.$$

We bound the first integral in the last line using the case $s=\frac{1}{2}$ we obtained above for $\widehat{G}_1(\xi,\lambda)=\xi^2\widehat{F}(\xi,\lambda)$, and the second integral using the proof of the case II for $\widehat{G}_2(\xi,\lambda)=\langle\lambda+\xi^2\rangle\widehat{F}(\xi,\lambda)$. Thus, we obtain

$$\Big\|\frac{d}{dt}\Big[\eta(t)D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')Fdt'\Big)\Big]\Big\|_{H^{\frac{1}{2}}}$$
$$\lesssim\|F\|_{X^{\frac{1}{2},-b}}+\|G_1\|_{X^{\frac{1}{2},-b}}+\|G_2\|_{X^{\frac{1}{2},-b}}\lesssim\|F\|_{X^{\frac{1}{2},1-b}}+\|F\|_{X^{\frac{5}{2},-b}},$$

for all $b<\frac{1}{2}$.

Therefore, we have

$$\Big\|\eta D_0\Big(\int_0^t W_{\mathbb{R}}(t-t')Fdt'\Big)\Big\|_{H^{\frac{2s+1}{4}}(\mathbb{R})}\lesssim\begin{cases}\|F\|_{X^{s,-b}} & \text{for } 0\leq s\leq\frac{1}{2}, b<\frac{1}{2},\\ \|F\|_{X^{\frac{1}{2},1-b}}+\|F\|_{X^{\frac{5}{2},-b}} & \text{for } s=\frac{5}{2}, b<\frac{1}{2}.\end{cases}$$

We obtain the statement for $\frac{1}{2}<s<\frac{5}{2}$ by interpolation.

$\square$

We now supply the nonlinear estimates that can close the argument.

Proposition 6.11. *For fixed $s > 0$ and $a < \min(2s, \frac{1}{2})$, there exists $\epsilon > 0$ such that for $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have*
$$\left\||u|^2 u\right\|_{X^{s+a,-b}} \lesssim \|u\|^3_{X^{s,b}}.$$

The proof the this proposition is analogues to Proposition 5.28. The fact that the integrals are based on the real line, makes the resonant case more interesting. We thus present the proof in full details.

Proof. By writing the Fourier transform of $|u|^2 u = u\bar{u}u$ as a convolution, we obtain
$$\widehat{|u|^2 u}(\xi,\tau) = \int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} \widehat{u}(\xi_1,\tau_1)\overline{\widehat{u}(\xi_2,\tau_2)}\widehat{u}(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2).$$
Hence
$$\||u|^2u\|^2_{X^{s+a,-b}} = \left\|\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} \frac{\langle\xi\rangle^{s+a}\widehat{u}(\xi_1,\tau_1)\overline{\widehat{u}(\xi_2,\tau_2)}\widehat{u}(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)}{\langle\tau+\xi^2\rangle^b}\right\|^2_{L^2_\xi L^2_\tau}.$$
We define
$$f(\xi,\tau) = |\widehat{u}(\xi,\tau)|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b$$
and
$$M(\xi_1,\xi_2,\xi,\tau_1,\tau_2,\tau) = \frac{\langle\xi\rangle^{s+a}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi-\xi_1+\xi_2\rangle^{-s}}{\langle\tau+\xi^2\rangle^b\langle\tau_1+\xi_1^2\rangle^b\langle\tau_2+\xi_2^2\rangle^b\langle\tau-\tau_1+\tau_2+(\xi-\xi_1+\xi_2)^2\rangle^b}.$$
It is then sufficient to show that
$$\left\|\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M(\xi_1,\xi_2,\xi,\tau_1,\tau_2,\tau)f(\xi_1,\tau_1)f(\xi_2,\tau_2)f(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\right\|^2_{L^2_\xi L^2_\tau}$$
$$\lesssim \|f\|^6_{L^2} = \|u\|^6_{X^{s,b}}.$$
By applying the Cauchy-Schwarz inequality in the $\xi_1,\xi_2,\tau_1,\tau_2$ integral and then using Hölder's inequality, we bound the norm above by
$$\left\|\left(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\right)^{1/2}\left(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f^2(\xi_1,\tau_1)f^2(\xi_2,\tau_2)f^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\right)^{1/2}\right\|^2_{L^2_\xi L^2_\tau}$$
$$= \left\|\left(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\right)\left(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f^2(\xi_1,\tau_1)f^2(\xi_2,\tau_2)f^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\right)\right\|_{L^1_\xi L^1_\tau}$$
$$\le \sup_{\xi,\tau}\left(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\right)\cdot\left\|\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f^2(\xi_1,\tau_1)f^2(\xi_2,\tau_2)f^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\right\|_{L^1_\xi L^1_\tau}$$
$$= \sup_{\xi,\tau}\left(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\right)\cdot\left\|f^2 * f^2 * f^2\right\|_{L^1_\xi L^1_\tau}.$$
Using Young's inequality, the norm $\left\|f^2 * f^2 * f^2\right\|_{L^1_\xi L^1_\tau}$ can be estimated by $\|f\|^6_{L^2_\xi L^2_\tau}$. Thus it is sufficient to show that the supremum above is finite. Integrating the τ_1,τ_2 integrals as before, we have that the supremum is bounded by
$$\sup_{\xi,\tau}\int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\tau+\xi^2\rangle^{2b}\langle\tau+\xi_1^2-\xi_2^2+(\xi-\xi_1+\xi_2)^2\rangle^{6b-2}}d\xi_1 d\xi_2.$$

Using the relation $\langle \tau - a\rangle\langle \tau - b\rangle \gtrsim \langle a-b\rangle$, the above reduces to

$$\sup_\xi \int \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle \xi^2-\xi_1^2+\xi_2^2-(\xi-\xi_1+\xi_2)^2\rangle^{1-}}d\xi_1 d\xi_2$$

$$= \sup_\xi \int \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle 2(\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2.$$

We break the integral into two pieces. The argument given in [27] (see also the proof of Proposition 5.28) shows that

$$\sup_\xi \int_{\substack{|\xi_1-\xi|\geq 1\\ |\xi_1-\xi_2|\geq 1}} \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle (\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2 < \infty.$$

To estimate the integral on the remaining set, $\{|\xi_1-\xi|\leq 1 \text{ or } |\xi_1-\xi_2|\leq 1\}$, note that

$$\langle \xi_1\rangle\langle \xi-\xi_1+\xi_2\rangle \sim \langle \xi_2\rangle\langle \xi\rangle. \tag{6.17}$$

Therefore, we have

$$\int_{\substack{|\xi_1-\xi|\leq 1 \text{ or}\\ |\xi_1-\xi_2|\leq 1}} \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle (\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2 \lesssim \int \frac{\langle \xi\rangle^{2a}\langle \xi_2\rangle^{-4s}}{\langle (\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2$$

we use the substitution $x=(\xi_1-\xi)(\xi_1-\xi_2)$ in the ξ_1 integral. This yields

$$2\xi_1 = \xi+\xi_2 \pm \sqrt{(\xi+\xi_2)^2-4(\xi\xi_2-x)} = \xi+\xi_2 \pm \sqrt{4x+(\xi-\xi_2)^2}$$

and

$$dx = (2\xi_1-\xi-\xi_2)\, d\xi_1 = \pm\sqrt{4x+(\xi-\xi_2)^2}\, d\xi_1.$$

Therefore, the integral above is bounded by

$$\int \frac{\langle \xi\rangle^{2a}\langle \xi_2\rangle^{-4s}}{\langle x\rangle^{1-}\sqrt{|4x+(\xi-\xi_2)^2|}}dx\, d\xi_2.$$

Using Lemma 6.9 and then Lemma 6.8 again, we bound the supremum of the integral above by

$$\sup_\xi \int \frac{\langle \xi\rangle^{2a}\langle \xi_2\rangle^{-4s}}{\langle (\xi-\xi_2)^2\rangle^{\frac12-}}d\xi_2 \lesssim \sup_\xi \int \frac{\langle \xi\rangle^{2a}\langle \xi_2\rangle^{-4s}}{\langle \xi-\xi_2\rangle^{1-}}d\xi_2$$

$$\lesssim \sup_\xi \begin{cases} \langle \xi\rangle^{2a-1+} & \text{for } s\geq \frac14 \\ \langle \xi\rangle^{2a-4s+} & \text{for } s<\frac14 \end{cases}.$$

For $a<\min(\frac12, 2s)$, this is finite. □

The following is the major proposition of this section:

Proposition 6.12. *For fixed $0<s<\frac52$, and $0\leq a<\min(2s,\frac12,\frac52-s)$, there exists $\epsilon>0$ such that for $\frac12-\epsilon<b<\frac12$, we have*

$$\text{for } 0<s+a\leq \frac12, \quad \big\||u|^2u\big\|_{X^{s+a,-b}} \lesssim \|u\|^3_{X^{s,b}},$$

$$\text{for } \frac12<s+a<\frac52, \quad \big\||u|^2u\big\|_{X^{\frac12,\frac{2s+2a-1-4b}{4}}} \lesssim \|u\|^3_{X^{s,b}}.$$

Proof. For $s + a \leq \frac{1}{2}$, the statement follows from Proposition 6.11.

We now consider the case $\frac{1}{2} < s + a < \frac{5}{2}$. Since $a < 2s$, we always have $s > \frac{1}{6}$. Let

$$S := \int \frac{\langle \tau + \xi^2 \rangle^{s+a-2b-\frac{1}{2}} \langle \xi \rangle \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \tau + \xi_1^2 - \xi_2^2 + (\xi - \xi_1 + \xi_2)^2 \rangle^{6b-2}} d\xi_1 d\xi_2.$$

Following the proof of Proposition 6.11, it suffices to prove that

$$\sup_{\xi, \tau} S < \infty.$$

We consider the cases $\frac{1}{2} < s + a < \frac{3}{2}$ and $\frac{3}{2} \leq s + a < \frac{5}{2}$ separately.

Case 1) $\frac{1}{2} < s + a < \frac{3}{2}$. Taking ϵ sufficiently small, we have $s + a - 2b - \frac{1}{2} < 0$. Using the identity $\langle \tau - a \rangle \langle \tau - b \rangle \gtrsim \langle a - b \rangle$, and noting that $2b + \frac{1}{2} - s - a < 6b - 2$ (for ϵ sufficiently small), we obtain

$$\begin{aligned} S &\lesssim \int \frac{\langle \xi \rangle \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \xi^2 - \xi_1^2 + \xi_2^2 - (\xi - \xi_1 + \xi_2)^2 \rangle^{2b+\frac{1}{2}-s-a}} d\xi_1 d\xi_2 \\ &\lesssim \int \frac{\langle \xi \rangle \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b+\frac{1}{2}-s-a}} d\xi_1 d\xi_2. \end{aligned}$$

We can estimate this for $s > \frac{1}{2}$ by

$$\int \langle \xi \rangle \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s} d\xi_1 d\xi_2 \lesssim 1$$

by using Lemma 6.8 twice.

It remains to consider the case $\frac{1}{6} < s \leq \frac{1}{2}$. Since $a < \min(2s, \frac{1}{2})$, we have $\frac{1}{2} < s + a < \min(3s, s + \frac{1}{2})$.

Consider the sets $A = \{|x_1 - \xi| < 1 \text{ or } |x_1 - \xi_2| < 1\}$ and $B = \{|x_1 - \xi| \geq 1 \text{ and } |x_1 - \xi_2| \geq 1\}$. Since on A we have (6.17), we obtain

$$\int_A \frac{\langle \xi \rangle \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b+\frac{1}{2}-s-a}} d\xi_1 d\xi_2 \lesssim \int_A \frac{\langle \xi \rangle^{1-2s} \langle \xi_2 \rangle^{-4s}}{\langle (\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b+\frac{1}{2}-s-a}} d\xi_1 d\xi_2.$$

Proceeding as in Proposition 6.11 by substituting $x = (\xi_1 - \xi)(\xi_1 - \xi_2)$ in the ξ_1 integral, we bound this by

$$\int \frac{\langle \xi \rangle^{1-2s} \langle \xi_2 \rangle^{-4s}}{\langle x \rangle^{2b+\frac{1}{2}-s-a} \sqrt{|4x + (\xi - \xi_2)^2|}} dx d\xi_2 \lesssim \int \frac{\langle \xi \rangle^{1-2s} \langle \xi_2 \rangle^{-4s}}{\langle \xi - \xi_2 \rangle^{2(2b-s-a)}} d\xi_2,$$

where we used Lemma 6.9 (taking ϵ sufficiently small). Using Lemma 6.8 (noting that $2(2b - s - a) < 1$), we bound this by

$$\begin{cases} \langle \xi \rangle^{2-4b+2a-4s+} & \text{for } s \leq \frac{1}{4} \\ \langle \xi \rangle^{1-4b+2a} & \text{for } s > \frac{1}{4} \end{cases}$$

which is bounded for $a < \min(2s, \frac{1}{2})$, provided that ϵ is sufficiently small.

We bound the integral on the set B by (after the change of variable $\xi_2 \to \xi_1+\xi_2$, $\xi_1 \to \xi+\xi_1$)

$$\int \frac{\langle\xi\rangle\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1-\xi\rangle^{2b+\frac{1}{2}-s-a}\langle\xi_1-\xi_2\rangle^{2b+\frac{1}{2}-s-a}}d\xi_1 d\xi_2 = \int \frac{\langle\xi\rangle\langle\xi+\xi_1\rangle^{-2s}\langle\xi+\xi_2\rangle^{-2s}\langle\xi+\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1\rangle^{2b+\frac{1}{2}-s-a}\langle\xi_2\rangle^{2b+\frac{1}{2}-s-a}}d\xi_1 d\xi_2.$$

By symmetry, we have the following subcases $|\xi+\xi_1+\xi_2| \gtrsim |\xi|$ and $|\xi+\xi_1| \gtrsim |\xi|$, which leads to the bound (using Lemma 6.8 repeatedly)

$$\begin{aligned}
&\langle\xi\rangle^{1-2s}\Big(\int \frac{\langle\xi+\xi_1\rangle^{-2s}}{\langle\xi_1\rangle^{2b+\frac{1}{2}-s-a}}d\xi_1\Big)^2 + \langle\xi\rangle^{1-2s}\int \frac{\langle\xi+\xi_2\rangle^{-2s}\langle\xi+\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1\rangle^{2b+\frac{1}{2}-s-a}\langle\xi_2\rangle^{2b+\frac{1}{2}-s-a}}d\xi_1 d\xi_2\\
&\lesssim \langle\xi\rangle^{1-2s}\big(\langle\xi\rangle^{-(s+2b-\frac{1}{2}-a)+}\big)^2 + \langle\xi\rangle^{1-2s}\int \frac{1}{\langle\xi+\xi_2\rangle^{3s+2b-\frac{1}{2}-a-}\langle\xi_2\rangle^{2b+\frac{1}{2}-s-a}}d\xi_2\\
&\lesssim \langle\xi\rangle^{2-4b+2a-4s+} + \begin{cases}\langle\xi\rangle^{2-4b+2a-4s+} & \text{for } 3s+2b-\frac{1}{2}-a\le 1\\ \langle\xi\rangle^{\frac{1}{2}-2b-s+a} & \text{for } 3s+2b-\frac{1}{2}-a>1\end{cases}
\end{aligned}$$

This is bounded for $a<\min(2s,\frac{1}{2})$, provided that ϵ is sufficiently small.

Case 2) $\frac{3}{2}\le s+a<\frac{5}{2}$. In this case $s+a-2b-\frac{1}{2}\ge 0$. Using

$$\begin{aligned}
\langle\tau+\xi^2\rangle &= \langle\tau+\xi_1^2-\xi_2^2+(\xi-\xi_1+\xi_2)^2+2(\xi-\xi_1)(\xi_1-\xi_2)\rangle\\
&\lesssim \langle\tau+\xi_1^2-\xi_2^2+(\xi-\xi_1+\xi_2)^2\rangle+\langle\xi-\xi_1\rangle\langle\xi_1-\xi_2\rangle.
\end{aligned}$$

Also noting that in this case $s+a-2b-\frac{1}{2}<6b-2$ for (ϵ sufficiently small), we have

$$\begin{aligned}
S &\lesssim \int \langle\xi-\xi_1\rangle^{s+a-2b-\frac{1}{2}}\langle\xi_1-\xi_2\rangle^{s+a-2b-\frac{1}{2}}\langle\xi\rangle\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}d\xi_1 d\xi_2\\
&= \int \langle\xi_1\rangle^{s+a-2b-\frac{1}{2}}\langle\xi_2\rangle^{s+a-2b-\frac{1}{2}}\langle\xi\rangle\langle\xi+\xi_1\rangle^{-2s}\langle\xi+\xi_2\rangle^{-2s}\langle\xi+\xi_1+\xi_2\rangle^{-2s}d\xi_1 d\xi_2.
\end{aligned}$$

Here we applied the change of variable $\xi_2\to\xi_1+\xi_2$, $\xi_1\to\xi+\xi_1$. Considering the subcases $|\xi+\xi_1+\xi_2|\gtrsim|\xi|$ and $|\xi+\xi_1|\gtrsim|\xi|$ we have the bound

$$\begin{aligned}
S \lesssim\ &\langle\xi\rangle^{1-2s}\Big(\int\langle\xi_1\rangle^{s+a-2b-\frac{1}{2}}\langle\xi+\xi_1\rangle^{-2s}d\xi_1\Big)^2\\
&+\langle\xi\rangle^{1-2s}\int\langle\xi_1\rangle^{s+a-2b-\frac{1}{2}}\langle\xi_2\rangle^{s+a-2b-\frac{1}{2}}\langle\xi+\xi_2\rangle^{-2s}\langle\xi+\xi_1+\xi_2\rangle^{-2s}d\xi_1 d\xi_2\\
&=: S_1+S_2.
\end{aligned}$$

Using $\langle\xi_1\rangle\lesssim\langle\xi+\xi_1\rangle\langle\xi\rangle$, we have

$$S_1\lesssim\langle\xi\rangle^{2a-4b}\Big(\int\langle\xi+\xi_1\rangle^{-s+a-2b-\frac{1}{2}}d\xi_1\Big)^2\lesssim 1$$

by the restrictions on a,b,s. Using $\langle\xi_1\rangle\lesssim\langle\xi+\xi_2\rangle\langle\xi+\xi_1+\xi_2\rangle$ and $\langle\xi_2\rangle\lesssim\langle\xi\rangle\langle\xi+\xi_2\rangle$ we have

$$S_2\lesssim\langle\xi\rangle^{\frac{1}{2}-s+a-2b}\int\langle\xi+\xi_2\rangle^{2a-4b-1}\langle\xi+\xi_1+\xi_2\rangle^{a-2b-\frac{1}{2}-s}d\xi_1 d\xi_2\lesssim 1$$

by the restrictions on a,b,s. $\square$

6.3. **Local theory: The proof of Theorem 6.2.** We first prove that

$$\Gamma u(t) := \eta(t) W_{\mathbb{R}}(t) g_e + \eta(t) \int_0^t W_{\mathbb{R}}(t-t') F(u)\, dt' + \eta(t) W_0^t\big(0, h-p-q\big)(t), \quad (6.18)$$

has a fixed point in $X^{s,b}$. Here $s \in (0, \frac{5}{2})$, $s \neq \frac{1}{2}, \frac{5}{2}$, $b < \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$, and

$$F(u) = \eta(t/T)|u|^2 u, \quad p(t) = \eta(t) D_0(W_{\mathbb{R}} g_e), \quad \text{and}$$
$$q(t) = \eta(t) D_0\Big(\int_0^t W_{\mathbb{R}}(t-t') F(u)\, dt'\Big).$$

To see that Γ is bounded in $X^{s,b}$ recall the following bounds:

By (6.13), we have

$$\|\eta W_{\mathbb{R}}(t) g_e\|_{X^{s,b}} \lesssim \|g_e\|_{H^s} \lesssim \|g\|_{H^s(\mathbb{R}^+)}.$$

Combining (6.14), (6.15), and Proposition 6.11, we obtain

$$\|\eta(t) \int_0^t W_{\mathbb{R}}(t-t') F(u)\, dt'\|_{X^{s,b}} \lesssim \|F(u)\|_{X^{s,-\frac{1}{2}+}} \lesssim T^{\frac{1}{2}-b-} \||u|^2 u\|_{X^{s,-b}} \lesssim T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}}.$$

Using Proposition 6.7 and Lemma 6.3 (noting that the compatibility condition holds) we have

$$\|\eta(t) W_0^t\big(0, h-p-q\big)(t)\|_{X^{s,b}} \lesssim \|(h-p-q)\chi_{(0,\infty)}\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}$$
$$\lesssim \|h-p\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|q\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} \lesssim \|h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|p\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} + \|q\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}. \quad (6.19)$$

By Kato smoothing Lemma 6.5, we have

$$\|p\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}.$$

Finally, by Propostion 6.10, (6.15), and Proposition 6.12 we have

$$\|q\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim \begin{cases} \|F\|_{X^{s,-\frac{1}{2}+}} & \text{for } 0 \leq s \leq \frac{1}{2} \\ \|F\|_{X^{\frac{1}{2}, \frac{2s-3+}{4}}} + \|F\|_{X^{s,-\frac{1}{2}+}} & \text{for } \frac{1}{2} < s < \frac{5}{2} \end{cases}$$
$$\lesssim T^{\frac{1}{2}-b-} \begin{cases} \||u|^2 u\|_{X^{s,-b}} & \text{for } 0 \leq s \leq \frac{1}{2} \\ \||u|^2 u\|_{X^{\frac{1}{2}, \frac{2s-1-4b}{4}}} + \||u|^2 u\|_{X^{s,-b}} & \text{for } \frac{1}{2} < s < \frac{5}{2} \end{cases} \lesssim T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}}.$$

Combining these estimates, we obtain

$$\|\Gamma u\|_{X^{s,b}} \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}}.$$

This yields the existence of a fixed point u in $X^{s,b}$. Now we prove that $u \in C_t^0 H_x^s([0,T) \times \mathbb{R})$. Note that the first term in the definition (6.18) is continuous in H^s. The continuity of the third term follows from Lemma 6.6 and (6.19). For the second term it follows from the embedding $X^{s,\frac{1}{2}+} \subset C_t^0 H_x^s$ and (6.14) together with Proposition 6.11. The fact that $u \in C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R} \times [0,T])$ follows similarly from Lemma 6.5, Proposition 6.10, and Lemma 6.6.

The continuous dependence on the initial and boundary data follows from the fixed point argument and the a priori estimates as in the previous paragraph. The uniqueness property is based on a priori energy bounds that one can derive for the solutions and the smoothing estimates we have established. For the details see [24] and [19].

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