Penrose stability analysis and highly localized instabilities of noisy, nonlinear wavefields

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The study of extreme waves in irregular unidirectional solutions of the Nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \frac{p}{2}\Delta u + q|u|^2 u = 0 \tag{1}$$

arises as a crucial question in many contexts. Recent results have demonstrated that random localizations of energy, induced by the linear dispersive mixing of different harmonics, can grow significantly due to nonlinear focusing effects. This process is widely thought to be responsible for the formation in Rogue Waves.

In this work we perform a linear stability analysis similar to the derivation of the modulation instability, but starting from a general Fourier spectrum $P(k)$, i.e. a general distribution of energy over wavenumbers. Using the Wigner transform and an adaptation of Penrose's method, we compute linearly unstable, spatially periodic modes, which we call *unstable Penrose modes*, and which can be thought of as the generalization of Benjamin-Feir sidebands. More specifically, for any $\zeta \in \mathbb{R}$, $\Omega \in \mathbb{C}$ satisfying the *Penrose condition*

$$\int\limits_{k\in\mathbb{R}} \frac{P(k-\frac{\zeta}{4\pi}) - P(k+\frac{\zeta}{4\pi})}{pk-\Omega} dk = \frac{2\pi\zeta}{q}, \tag{2}$$

there is an unstable mode of wavenumber ζ, exponential rate of growth $\zeta \operatorname{Im}\Omega$ and speed of propagation $\operatorname{Re}\Omega$. Furthermore, carefully analysing integrals of unstable Penrose modes, we recover *persistently localized unstable modes.*

This computation associates with every observed distribution of energy per wavenumber, $P(k)$, the shortest lengthscale over which coherent structures can form, and the timescale for which their evolution is well approximated by our method. Thus a criterion for whether a given spectrum supports rogue waves is formulated, and subsequently applied to realistic Fourier spectra of ocean waves.

Moreover, the case of unimodal spectra even around their maximum is fully characterised analytically in terms of the existence of Penrose modes and their sharpest possible localisation.

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